UNIVERSITY OF ANBAR
COLLEGE OF ENGINEERING
DAM & WATER RESOURCES

STRENGTH OF MATERIALS

Dams &Water Resources Engineering Department
Second Stage – 2st Semester
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Lecture Notes Prepared by:

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Course Description

The course named " Strength of Materials" or "Mechanics of Materials" deals with, Concept of stress, Stresses and strains, Axial loading and axial deformation, Hook's law, Statically indeterminate members, Stresses due to temperature, Torsion, Internal forces in beams, pure bending or Beam theory, Transverse loading and shear stresses in beams, beam deflection, Transformation of stresses and strains,. Principal stresses and strains, in addition to Axially compressed members and buckling of columns.

Course Objectives

1. Be aware of the mathematical background for the different topics of strength of materials introduced in this course.
2. Understanding of stress concept and types of stresses.
3. Understanding of stress strain relationship and solving problems.
4. Understanding of internal forces in beams, how to draw shear force and bending moment diagrams.
5. Understanding of beam analysis, stresses in beams, beam theory and shear stresses.
6. Understanding of torsion in shafts, determination of shear stresses and twisting angle due to torsion.
7. Understanding of methods of calculation beam deflection.
8. Understanding of transformation of stresses and constructing of Mohr's Circle.
9. Understanding of Axially compressed members and buckling of columns.

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TOPICS

CHAPTER 1

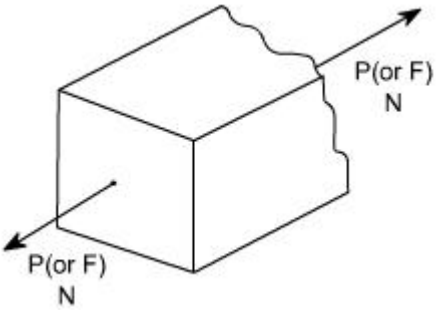
Stress

Concept of Stress : Let us introduce the concept of stress, as we know that the main problem of engineering mechanics of material is the investigation of the internal resistance of the body, i.e. the nature of forces set up within a body to balance the effect of the externally applied forces.

The externally applied forces are termed as loads. These externally applied forces may be due to any one or more of the followings:

(i) due to service conditions
(ii) due to environment in which the component works
(iii) through contact with other members
(iv) due to fluid pressures
(v) due to gravity or inertia forces (Self weight of the structure).

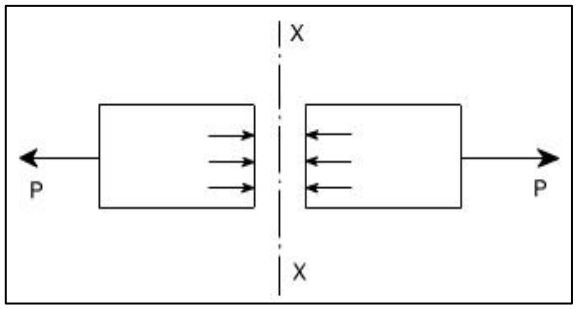
As we know that in mechanics of deformable solids, externally applied forces acts on a body and body suffers a deformation. From equilibrium point of view, this action should be opposed or reacted by internal forces which are set up within the particles of material due to cohesion. These internal forces give rise to a concept of stress. Therefore, let us define a term stress:



Stress:

Let us consider a rectangular bar of some cross–sectional area and subjected to some load or force (in Newton).

Let us imagine that the same rectangular bar is assumed to be cut into two halves at section XX. Each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section XX has been shown.



Simple Stress

Simple stress is expressed as the ratio of the applied force divided by the resisting area or :

$$\sigma = \text{Force} / \text{Area}.$$

It is the expression of force per unit area to structural members that are subjected to external forces and/or induced forces. Here we are using an assumption that the total force or total load carried by the bar is uniformly distributed over its cross_section.

Units :

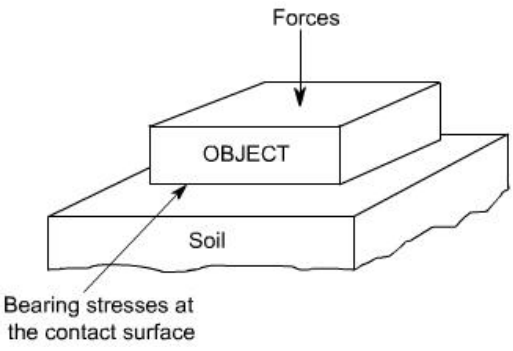
The basic units of stress in S.I units i.e. (International System) are **N / m^2** (or Pa , Pascal)

MPa = 10^6 Pa , GPa = 10^9 Pa , KPa = 10^3 Pa

Sometimes **N/mm^2** units are also used, because this is an equivalent to MPa , while **US** customary unit is ***pound per square*** inch , **psi**. (lb/in^2).

Simple stress can be classified as **normal stress**, **shear stress**, **and bearing stress**. **Normal stress** develops when a force is applied perpendicular to the cross-sectional area of the material. If the force is going to pull the material, the stress is said to be **tensile stress** and **compressive stress** develops when the material is being compressed by two opposing forces.

Shear stress is developed if the applied force is parallel to the resisting area. Example is the bolt that holds the tension rod in its anchor. Another condition of shearing is when we twist a bar along its longitudinal axis. This type of shearing is called torsion and covered in Chapter 3.



Another type of simple stress is the **bearing stress**, it is the contact pressure between two bodies. (It is in fact a compressive stress).

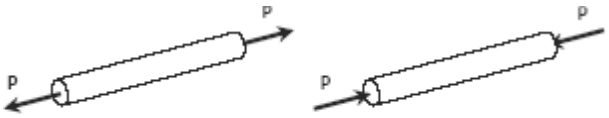
Suspension bridges are good example of structures that carry these stresses. The weight of the vehicle is carried by the bridge deck and passes the force to the stringers (vertical cables), which in turn, supported by the main suspension cables. The suspension cables then transferred the force into bridge towers.

Normal Stress

The resisting area is perpendicular to the applied force, thus normal. There are two types of normal stresses; tensile stress and compressive stress. Tensile stress applied to bar tends the bar to elongate while compressive stress tend to shorten the bar.

σ = Force / Area

σ = P/ A



Bar in Tension Bar in Compression

where P is the applied normal load in Newton and A is the area in mm^2. The maximum stress in tension or compression occurs over a section normal to the load.

EXAMPLE PROBLEMS IN NORMAL STRESS

Example 101: A hollow steel tube with an inside diameter of 100 mm must carry a tensile load of 400 kN. Determine the outside diameter of the tube if the stress is limited to 120 MN/m^2.

Solution 101:

$P = \sigma A$

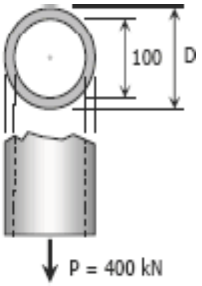
where:

$P = 400\ kN = 400\ 000\ N$

$\sigma = 120\ MPa$

$A = \frac{1}{4}\pi D^2 - \frac{1}{4}\pi(100^2)$

$= \frac{1}{4}\pi(D^2 - 10\ 000)$



thus,

$400\ 000 = 120[\frac{1}{4}\pi(D^2 - 10\ 000)]$

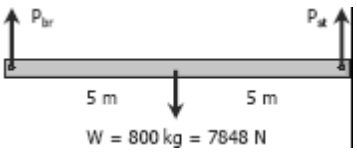
$400\ 000 = 30\pi D^2 - 300\ 000\pi$

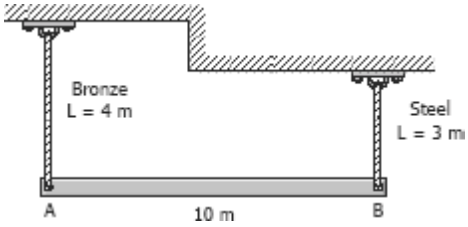
$D^2 = \frac{400\ 000 + 300\ 000\pi}{30\pi}$

$D = 119.35\ mm$

Example 102 A homogeneous 800 kg bar AB is supported at either end by a cable as shown in Fig. P-105. Calculate the smallest area of each cable if the stress is not to exceed 90 MPa in bronze and 120 MPa in steel.

Solution:





By symmetry:

$$P_{br} = P_{st} = \tfrac{1}{2}(7848)$$
$$= 3924 \text{ N}$$

For bronze cable:

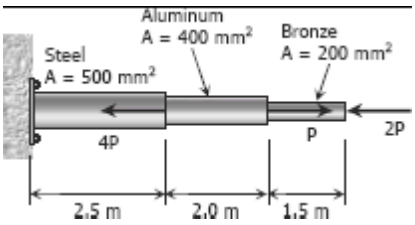
$$P_{br} = \sigma_{br} A_{br}$$
$$3924 = 90A_{br}$$
$$A_{br} = 43.6 \text{ mm}^2$$

For steel cable:

$$P_{st} = \sigma_{st} A_{st}$$
$$3924 = 120A_{st}$$
$$A_{st} = 32.7 \text{ mm}^2$$

Example 103 An aluminum rod is rigidly attached between a steel rod and a bronze rod as shown in Fig. P-108. Axial loads are applied at the positions indicated. Find the maximum value of P that will not exceed a stress in steel of 140 MPa, in aluminum of 90 MPa, or in bronze of 100 MPa.

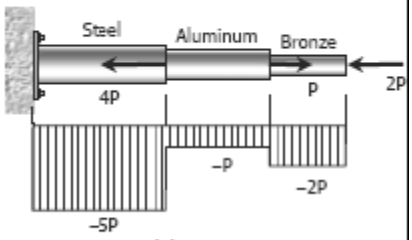
Solution:



For bronze:
$\sigma_{br} A_{br} = 2P$
$100(200) = 2P$
$P = 10\,000$ N

For aluminum:
$\sigma_{al} A_{al} = P$
$90(400) = P$
$P = 36\,000$ N

For Steel:
$\sigma_{st} A_{st} = 5P$
$P = 14\,000$ N



For safe P, use $P = 10\,000$ N = **10 kN**

Example 104 A 12-inches square steel bearing plate lies between an 8-inches diameter wooden post and a concrete footing as shown in Fig. P-110. Determine the maximum value of the load P if the stress in wood is limited to 1800 psi and that in concrete to 650 psi.

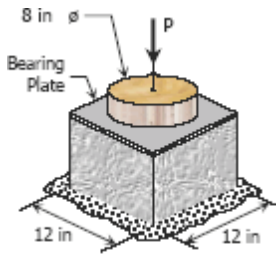


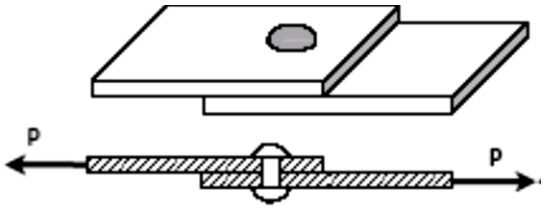
Figure P-110

Shearing Stress

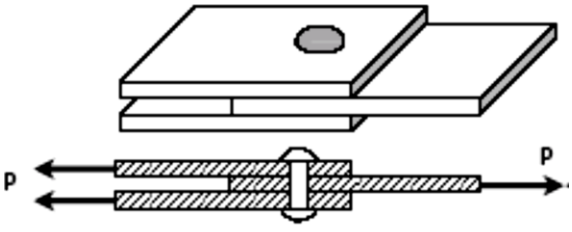
Forces parallel to the area resisting the force cause shearing stress. It differs to tensile and compressive stresses, which are caused by forces perpendicular to the area on which they act. Shearing stress is also known as tangential stress.

$$\tau = \frac{V}{A}$$

where V is the resultant shearing force which passes through the centroid of the area A being sheared.



Single Shear

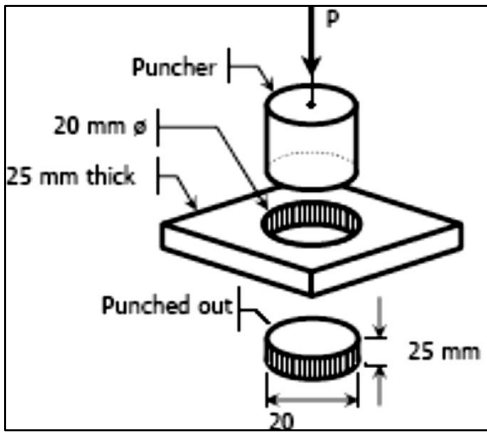


Double Shear

SOLVED EXAMPLES IN SHEARING STRESS

Example 105 : What force is required to punch a 20-mm-diameter hole in a plate that is 25 mm thick? The shear strength is 350 MN/m^2.

Solution:



The resisting area is the shaded area along the perimeter and the shear force V is equal to the punching force P.

$$V = \tau A$$
$$P = 350[\pi(20)(25)]$$
$$= 549\,778.7 \text{ N}$$
$$= \mathbf{549.8 \text{ kN}}$$

Example 106 Find the smallest diameter bolt that can be used in the clevis shown in Fig. 1-11b if P = 400 kN. The shearing strength of the bolt is 300 MPa.

Solution :

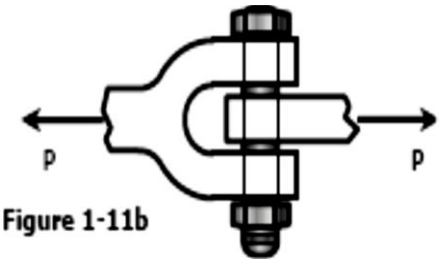


Figure 1-11b

The bolt is subject to double shear.

$$V = \tau A$$
$$400(1000) = 300[2(\tfrac{1}{4}\pi d^2)]$$
$$d = 29.13 \text{ mm}$$

Example 107 Compute the shearing stress in the pin at B for the member supported as shown in Fig. The pin diameter is 20 mm.

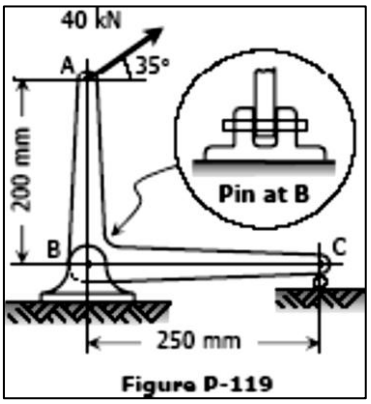
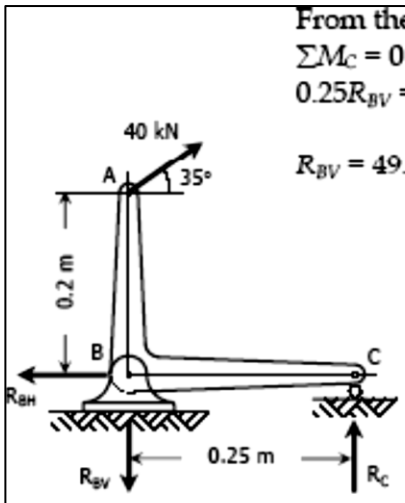


Figure P-119

Solution :



Free Body Diagram

From the FBD:

$\Sigma M_C = 0$

$0.25R_{BV} = 0.25(40 \sin 35°) + 0.2(40 \cos 35°)$

$R_{BV} = 49.156$ kN

$\Sigma F_H = 0$

$R_{BH} = 40 \cos 35°$

$= 32.766$ kN

$R_B = \sqrt{R_{BH}^2 + R_{BV}^2}$

$= \sqrt{32.766^2 + 49.156^2}$

$= 59.076$ kN → shear force of pin at B

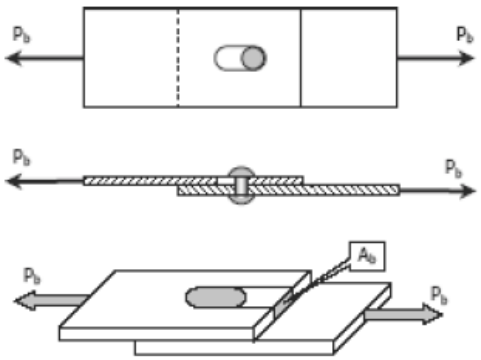
$V_B = \tau_B A$ → double shear

$59.076\ (1000) = \tau_B \{2[\frac{1}{4}\pi(20^2)]\}$

$\tau_B = \mathbf{94.02\ MPa}$

Bearing Stress

Bearing stress is the contact pressure between the separate bodies. It differs from compressive stress, as it is an internal stress caused by compressive forces.



$$\sigma_b = \frac{P_b}{A_b}$$

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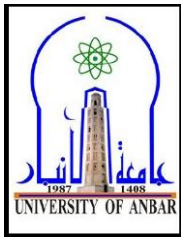


SOLVED EXAMPLES IN BEARING STRESS

Example 125 In Fig. 1-12, assume that a 20-mm-diameter rivet joins the plates that are each 110 mm wide. The allowable stresses are 120 MPa for bearing in the plate material and 60 MPa for shearing of rivet. Determine (a) the minimum thickness of each plate; and (b) the largest average tensile stress in the plates.

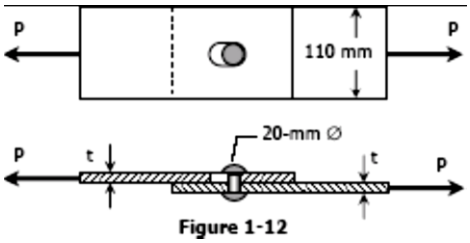


Figure 1-12

Solution

(a) From shearing of rivet:

$P = \tau A_{rivets}$

$= 60[\frac{1}{4}\pi(20^2)]$

$= 6000\pi$ N

From bearing of plate material:

$P = \sigma_b A_b$

$6000\pi = 120(20t)$

$t = 7.85$ mm

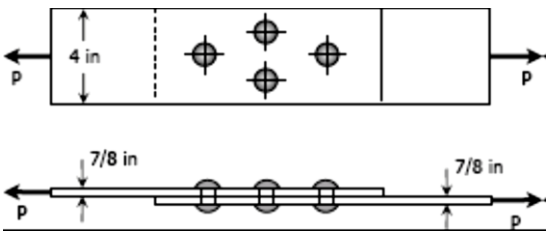
(b) Largest average tensile stress in the plate

$P = \sigma A$

$6000\pi = \sigma[7.85(110 - 20)]$

$\sigma = 26.67$ MPa

Example 126 The lap joint shown in Fig. P-126 is fastened by four ¾-in.-diameter rivets. Calculate the maximum safe load P that can be applied if the shearing stress in the rivets is limited to 14 ksi and the bearing stress in the plates is limited to 18 ksi. Assume the applied load is uniformly distributed among the four rivets.



Solution

Based on shearing of rivets:

$P = \tau A$

$P = 14[4\,(\frac{1}{4}\pi)(\frac{3}{4})^2\,]$

$P = 24.74$ kips

Based on bearing of plates:

$P = \sigma_b A_b$

$P = 18[4\,(\frac{3}{4})(\frac{7}{8})\,]$

$P = 47.25$ kips

Safe load P = **24.74 kips**

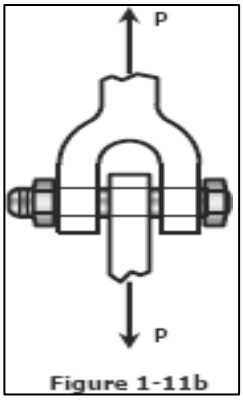
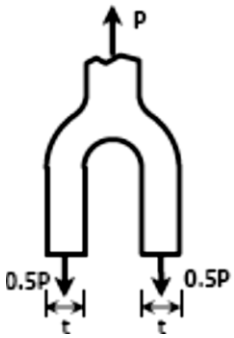


Figure 1-11b

Example 127: In the clevis shown in Fig. 1-11b, find the minimum bolt diameter and the minimum thickness of each yoke that will support a load P = 14 kips without exceeding a shearing stress of 12 ksi and a bearing stress of 20 ksi.

Solution:



For shearing of rivets (double shear)

$P = \tau A$

$14 = 12[2(\frac{1}{4}\pi d^2)\,]$

d = **0.8618 in** → diameter of bolt

For bearing of yoke:

$P = \sigma_b A_b$

$14 = 20[2(0.8618t)]$

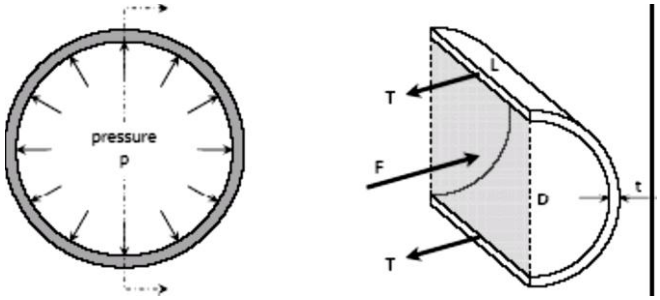
t = **0.4061 in** → thickness of yoke

Thin-Walled Pressure Vessels

A tank or pipe carrying a fluid or gas under a pressure is subjected to tensile forces, which resist bursting, developed across longitudinal and transverse sections.

Tangential Stress(Circumferential Stress):

Consider the tank shown being subjected to an internal pressure p. The length of the tank is L and the wall thickness is t. Isolating the right half of the tank:



$$F = pA = pDL$$

$$T = \sigma_t A_{wall} = \sigma_t tL$$

$$\sum F_H = 0$$

$$F = 2T$$

$$pDL = 2(\sigma_t tL)$$

$$\sigma_t = pD/2t$$

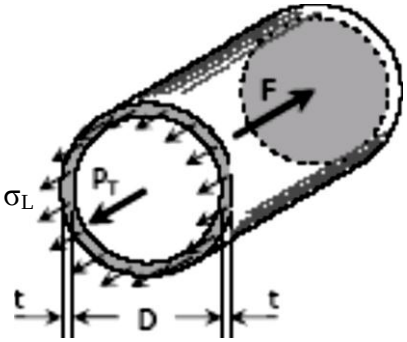
If there exist an external pressure p_o and an internal pressure p_i, the formula may be expressed as:

$$\sigma_t = \frac{(pi - po)D}{2t}$$

LONGITUDINAL STRESS, σ_L

Consider the free body diagram in the transverse section of the tank:

The total force acting at the rear of the tank F must equal to the total longitudinal stress on the wall $P_T = \sigma_L A_{wall}$. Since t is so small compared to D, the area of the wall is close to πDt .



$$F = pA = p\frac{\pi}{4}D^2$$

$$P_T = \sigma_L \pi Dt$$

$$[\Sigma F_H = 0]$$

$$P_T = F$$

$$\sigma_L \pi Dt = p\frac{\pi}{4}D^2$$

$$\sigma_L = \frac{pD}{4t}$$

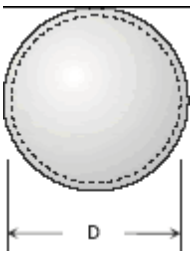
If there exist an external pressure p_o and an internal pressure p_i , the formula may be expressed as:

$$\sigma_L = \frac{(pi - po)D}{4t}$$

It can be observed that the tangential stress is twice that of the longitudinal stress.

$$\sigma_t = 2\,\sigma_L$$

Spherical Shell: If a spherical tank of diameter ***D*** and thickness ***t*** contains gas under a pressure of ***p***, the stress at the wall can be expressed as:

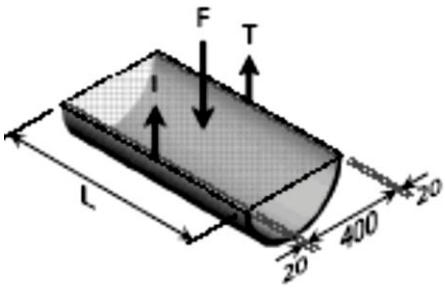


$$\sigma_L = \frac{(pi - po)D}{4t}$$

SOLVED EXAMPLES IN THIN WALLED PREASSURE VESSELS

Example 133: A cylindrical steel pressure vessel 400 mm in diameter with a wall thickness of 20 mm, is subjected to an internal pressure of 4.5 MN/m2. (a) Calculate the tangential and longitudinal stresses in the steel. (b) To what value may the internal pressure be increased if the stress in the steel is limited to 120 MN/m^2? (c) If the internal pressure were increased until the vessel burst, sketch the type of fracture that would occur.

Solution



Longitudinal Section

(a) Tangential stress (longitudinal section):

$$F = 2T$$

$$pDL = 2(\sigma_t\, tL)$$

$$\sigma_t = \frac{pD}{2t} = \frac{4.5(400)}{2(20)}$$

$$\sigma_t = 45 \text{ MPa}$$

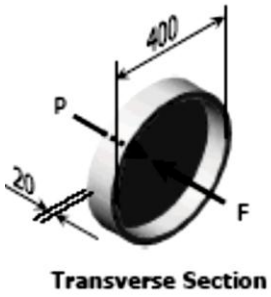
Longitudinal Stress (transverse section)::

$$F = P$$

$$\tfrac{1}{4}\pi D^2 p = \sigma_l\,(\pi D t)$$

$$\sigma_l = \frac{pD}{4t} = \frac{4.5(400)}{4(20)}$$

σ_l = **22.5 MPa**



Transverse Section

(b) From (a), $\sigma_t = \frac{pD}{2t}$ and $\sigma_l = \frac{pD}{4t}$ thus, $\sigma_t = 2\sigma_l$, this shows that tangential stress is the critical.

$$\sigma_t = \frac{pD}{2t}$$

$$120 = \frac{p(400)}{2(20)}$$

$$P = 12 \text{ MPa}$$

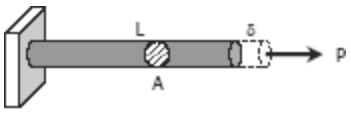
(c) The bursting force will cause a stress on the longitudinal section that is twice to that of the transverse section. Thus, fracture is expected as shown.

CHAPTER 2

STRAIN

Simple Strain

Strain (**ε**) is the ratio of the change in length caused by the applied force, to the original length.(Also known as unit deformation).



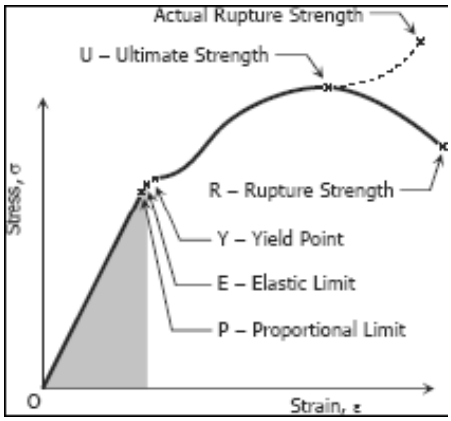
$$\varepsilon = \frac{\delta}{L}$$

where δ is the deformation and L is the original length, thus **ε** is dimensionless.

Stress-Strain Diagram

Suppose that a metal specimen be placed in tension-compression testing machine. As the axial load is gradually increased in increments, the total elongation over the gage length is measured at each increment of the load and this is continued until failure of the specimen takes place. Knowing the original cross-sectional area and length of the specimen, the normal stress σ and the strain ε can be obtained. The graph of these quantities with the stress σ along the y-axis and the strain ε along the x-axis is called the stress-strain diagram. The stress-strain diagram differs in form for various materials. The diagram shown below is that for a medium carbon structural steel.
Metallic engineering materials are classified as either ductile or brittle materials. A ductile material is one having relatively large tensile strains up to the point of rupture like structural steel and aluminum, whereas brittle

materials has a relatively small strain up to the point of rupture like cast iron and concrete. An arbitrary strain of 0.05 mm/mm is frequently taken as the dividing line between these two classes.



Proportional Limit (Hooke's Law)

From the origin o to the point called proportional limit, the stress-strain curve is a straight line. This linear relation between elongation and the axial force causing was first noticed by Sir Robert Hooke in 1678 and is called **Hooke's Law that within the proportional limit, the stress is directly proportional to strain or:**

$$\sigma \propto \varepsilon \ \ or$$
$$\sigma = k\,\varepsilon$$

The constant of proportionality **k** is called the Modulus of Elasticity **E** or Young's Modulus and is equal to the slope of the stress-strain diagram from O to P. Then :

$$\sigma = E\,\varepsilon$$

Elastic Limit

The elastic limit is the limit beyond which the material will no longer go back to its original shape when the load is removed, or it is the maximum stress that may be developed such that there is **nonpermanent** (or **residual**)

deformation when the load is entirely removed.

Elastic and Plastic Ranges

The region in stress-strain diagram from O to P is called the elastic range. The region from P to R is called the plastic range.

Yield Point

Yield point is the point at which the material will have an appreciable elongation or yielding without any increase in load.

Ultimate Strength

The maximum ordinate in the stress-strain diagram is the ultimate strength or tensile strength.

Rupture Strength

Rupture strength is the strength of the material at rupture. This is also known as the breaking strength.

Modulus Of Resilience

Modulus of resilience is the work done on a unit volume of material as the force is gradually increased from O to P, in Nm/m3. This may be calculated as the area under the stress-strain curve from the origin O to up to the elastic limit E (the shaded area in the figure). The resilience of the material is its ability to absorb energy without creating a permanent distortion.

Modulus Of Toughness

Modulus of toughness is the work done on a unit volume of material as the force is gradually increased from O to R, in Nm/m3. This may be calculated as the area under the entire stress-strain curve (from O to R). The toughness of a material is its ability to absorb energy without causing it to break.

STIFFNESS, k

Stiffness is the ratio of the steady force acting on an elastic body to the resulting displacement. It has the unit of N/mm.

$k = P / \delta$

Working Stress, Allowable Stress, And Factor Of Safety

Working stress is defined as the actual stress of a material under a given loading. The maximum safe stress that a material can carry is termed as the allowable stress. The allowable stress should be limited to values not exceeding the proportional limit. However, since proportional limit is difficult to determine accurately, the allowable tress is taken as either the yield point or ultimate strength divided by a factor of safety. The ratio of this strength (ultimate or yield strength) to allowable strength is called the factor of safety.

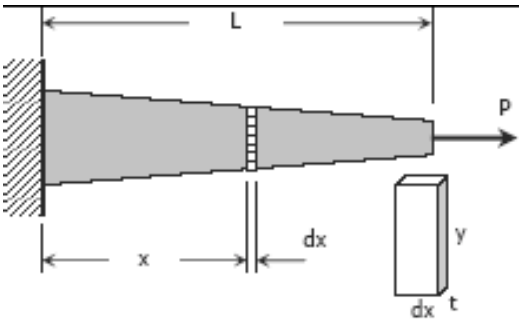
AXIAL DEFORMATION

In the linear portion of the stress-strain diagram, the tress is proportional to strain and is given by:$\boldsymbol{\sigma = E\varepsilon}$

since $\sigma = P / A$ and $\varepsilon = \delta / L$, then $P / A = E\,\delta / L$. Solving for δ,

$$\boldsymbol{\delta = \frac{PL}{AE} = \frac{\sigma L}{E}}$$

To use this formula, the load must be axial, the bar must have a uniform cross-sectional area, and the stress must not exceed the proportional limit. If however, the cross-sectional area is not uniform, the axial deformation can be determined by considering a differential length and applying integration.



where A = ty and y and t, if variable, must be

$$\delta = \frac{P}{E}\int_0^L \frac{dx}{A}$$

expressed in terms of x.

For a rod of unit mass ρ suspended vertically from one end, the total elongation due to its own weight is :

$$\delta = \frac{\rho g L^2}{2E} = \frac{MgL}{2AE}$$

where ρ is in kg/m^3, L is the length of the rod in mm, M is the total mass of the rod in kg, A is the cross-sectional area of the rod in mm^2, and g = 9.81 m/s^2.

SOLVED EXAMPLES ON STRAIN & AXIAL DEFORMATION

Concept Application 2.1

Determine the deformation of the steel rod shown in Fig. 2.19*a* under the given loads ($E = 29 \times 10^6$ psi).

The rod is divided into three component parts in Fig. 2.19*b*, so

$$L_1 = L_2 = 12 \text{ in.} \qquad L_3 = 16 \text{ in.}$$
$$A_1 = A_2 = 0.9 \text{ in}^2 \qquad A_3 = 0.3 \text{ in}^2$$

To find the internal forces P_1, P_2, and P_3, pass sections through each of the component parts, drawing each time the free-body diagram of the portion of rod located to the right of the section (Fig. 2.19*c*). Each of the free bodies is in equilibrium; thus

$$P_1 = 60 \text{ kips} = 60 \times 10^3 \text{ lb}$$
$$P_2 = -15 \text{ kips} = -15 \times 10^3 \text{ lb}$$
$$P_3 = 30 \text{ kips} = 30 \times 10^3 \text{ lb}$$

Using Eq. (2.10)

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)$$
$$= \frac{1}{29 \times 10^6}\left[\frac{(60 \times 10^3)(12)}{0.9} + \frac{(-15 \times 10^3)(12)}{0.9} + \frac{(30 \times 10^3)(16)}{0.3}\right]$$
$$\delta = \frac{2.20 \times 10^6}{29 \times 10^6} = 75.9 \times 10^{-3} \text{ in.}$$

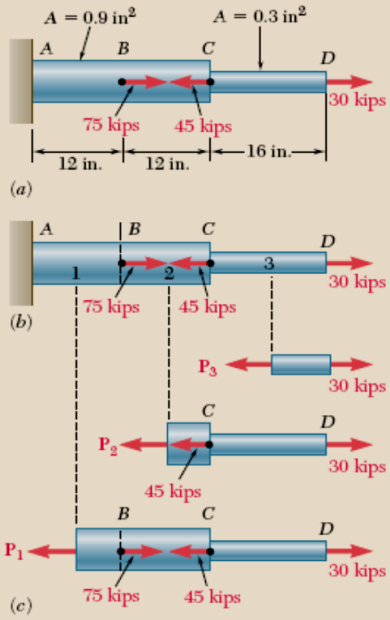
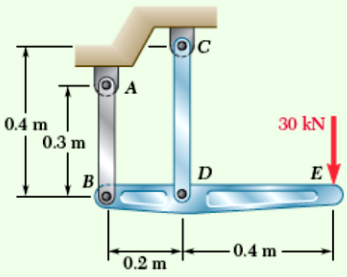


Fig. 2.19 (*a*) Axially-loaded rod. (*b*) Rod divided into three sections. (*c*) Three sectioned free-body diagrams with internal resultant forces P_1, P_2, and P_3.

Sample Problem 2.1



The rigid bar *BDE* is supported by two links *AB* and *CD*. Link *AB* is made of aluminum ($E = 70$ GPa) and has a cross-sectional area of 500 mm^2. Link *CD* is made of steel ($E = 200$ GPa) and has a cross-sectional area of 600 mm^2. For the 30-kN force shown, determine the deflection (*a*) of *B*, (*b*) of *D*, and (*c*) of *E*.

STRATEGY: Consider the free body of the rigid bar to determine the internal force of each link. Knowing these forces and the properties of the links, their deformations can be evaluated. You can then use simple geometry to determine the deflection of E.

MODELING: Draw the free body diagrams of the rigid bar (Fig. 1) and the two links (Fig. 2 and 3)

ANALYSIS:

Free Body: Bar BDE (Fig. 1)

$+\circlearrowleft \Sigma M_B = 0$: $\quad -(30\text{ kN})(0.6\text{ m}) + F_{CD}(0.2\text{ m}) = 0$

$F_{CD} = +90\text{ kN} \qquad F_{CD} = 90\text{ kN}$ *tension*

$+\circlearrowleft \Sigma M_D = 0$: $\quad -(30\text{ kN})(0.4\text{ m}) - F_{AB}(0.2\text{ m}) = 0$

$F_{AB} = -60\text{ kN} \qquad F_{AB} = 60\text{ kN}$ *compression*

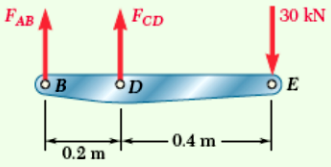


Fig. 1 Free-body diagram of rigid bar *BDE*.

a. Deflection of B. Since the internal force in link *AB* is compressive (Fig. 2), $P = -60$ kN and

$$\delta_B = \frac{PL}{AE} = \frac{(-60 \times 10^3\text{ N})(0.3\text{ m})}{(500 \times 10^{-6}\text{ m}^2)(70 \times 10^9\text{ Pa})} = -514 \times 10^{-6}\text{ m}$$

The negative sign indicates a contraction of member *AB*. Thus, the deflection of end *B* is upward:

$$\delta_B = 0.514\text{ mm} \uparrow$$

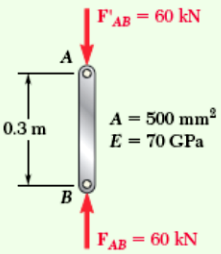


Fig. 2 Free-body diagram of two-force member *AB*.

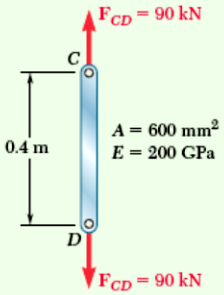


Fig. 3 Free-body diagram of two-force member *CD*.

(continued)

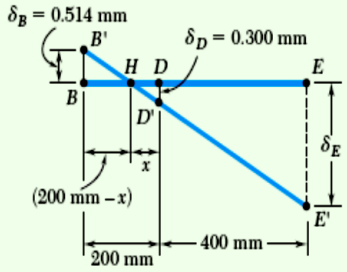


Fig. 4 Deflections at *B* and *D* of rigid bar are used to find δ_E.

b. Deflection of D. Since in rod *CD* (Fig. 3), $P = 90$ kN, write

$$\delta_D = \frac{PL}{AE} = \frac{(90 \times 10^3\ \text{N})(0.4\ \text{m})}{(600 \times 10^{-6}\ \text{m}^2)(200 \times 10^9\ \text{Pa})}$$

$$= 300 \times 10^{-6}\ \text{m} \qquad \delta_D = 0.300\ \text{mm} \downarrow$$

c. Deflection of E. Referring to Fig. 4, we denote by B' and D' the displaced positions of points B and D. Since the bar BDE is rigid, points B', D', and E' lie in a straight line. Therefore,

$$\frac{BB'}{DD'} = \frac{BH}{HD} \qquad \frac{0.514\ \text{mm}}{0.300\ \text{mm}} = \frac{(200\ \text{mm}) - x}{x} \qquad x = 73.7\ \text{mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD} \qquad \frac{\delta_E}{0.300\ \text{mm}} = \frac{(400\ \text{mm}) + (73.7\ \text{mm})}{73.7\ \text{mm}}$$

$$\delta_E = 1.928\ \text{mm} \downarrow$$

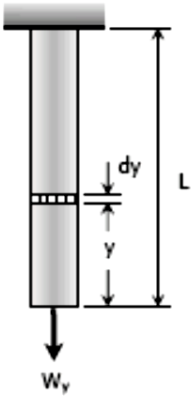
REFLECT and THINK: Comparing the relative magnitude and direction of the resulting deflections, you can see that the answers obtained are consistent with the loading and the deflection diagram of Fig. 4.

Example 201: A uniform bar of length L, cross-sectional area A, and unit mass ρ is suspended vertically from one end. Show that its total elongation is $\delta = \rho g L^2 / 2E$. If the total mass of the bar is M, show also that $\delta = MgL/2AE$.

Solution 201

$$\delta = \frac{PL}{AE}$$

From the figure:
$\delta = d\delta$
$P = W_y = (\rho A y)g$
$L = dy$



$$d\delta = \frac{(\rho A y)g\, dy}{AE}$$

$$\delta = \frac{\rho g}{E}\int_0^L y\, dy = \frac{\rho g}{E}\left[\frac{y^2}{2}\right]_0^L$$

$$\delta = \frac{\rho g}{2E}\left[L^2 - 0^2\right] = \rho g L^2/2E \qquad ok!$$

Given the total mass M:
$\rho = M/V = M/AL$

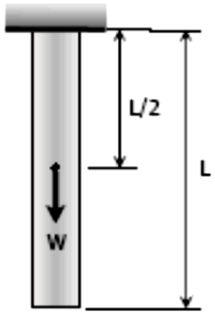
$\delta = \rho g L^2/2E = (M/AL)(gL^2/2E)$
$\delta = MgL/2AE$ ok!

Another Solution:

The weight will act at the center of gravity of the bar:

$$\delta = \frac{PL}{AE}$$

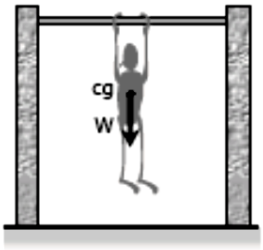
Where: $P = W = (\rho AL)g$
$L = L/2$



$$\delta = \frac{[(\rho AL)g](L/2)}{AE}$$

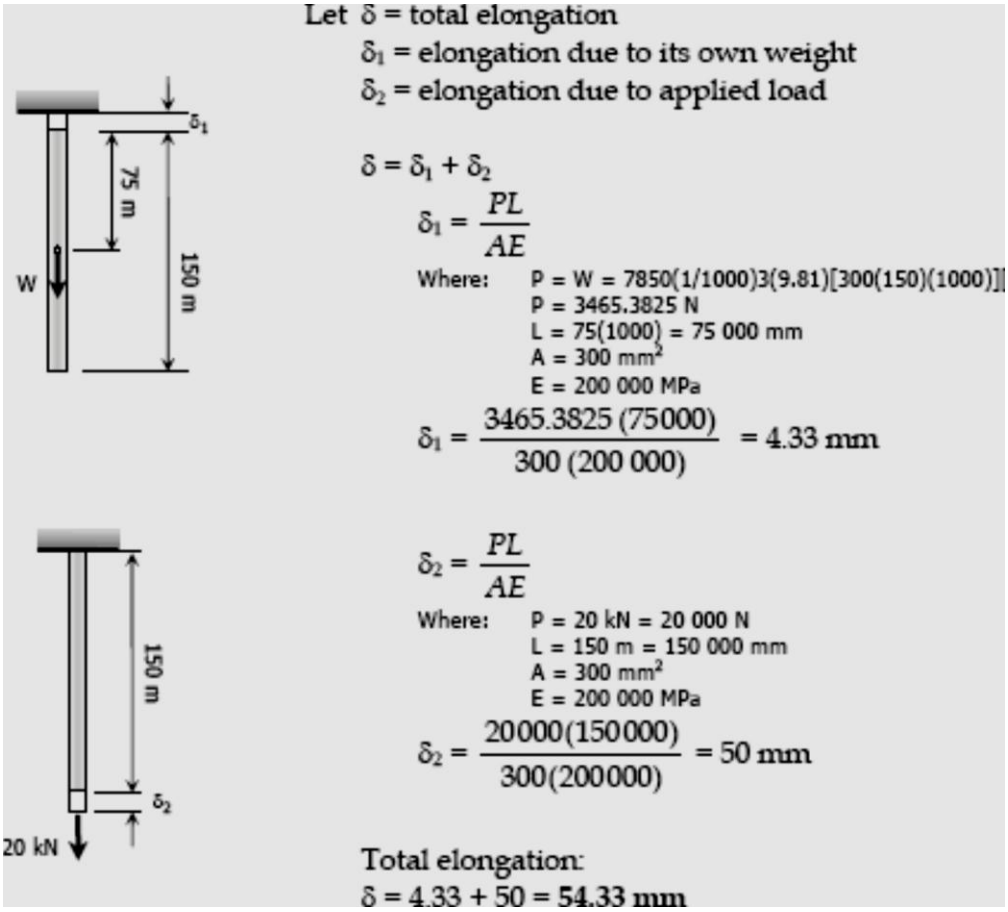
$$\delta = \frac{\rho g L^2}{2E} \qquad ok!$$

For you to feel the situation, position yourself in pull-up exercise with your hands on the bar and your body hang freely above the ground. Notice that your arms suffer all your weight and your lower body fells no stress (center of weight is approximately just below the chest). If your body is the bar, the elongation will occur at the upper half of it.



Example 201A :A steel rod having a cross-sectional area of 300 mm^2 and a length of 150 m is suspended vertically from one end. It supports a tensile load of 20 kN at the lower end. If the unit mass of steel is 7850 kg/m^3 and E = 200 × 10^3 MN/m^2, find the total elongation of the rod.

Solution 201A



Let δ = total elongation
δ_1 = elongation due to its own weight
δ_2 = elongation due to applied load

$\delta = \delta_1 + \delta_2$

$$\delta_1 = \frac{PL}{AE}$$

Where: P = W = 7850(1/1000)3(9.81)[300(150)(1000)]
P = 3465.3825 N
L = 75(1000) = 75 000 mm
A = 300 mm^2
E = 200 000 MPa

$$\delta_1 = \frac{3465.3825\,(75000)}{300\,(200\,000)} = 4.33 \text{ mm}$$

$$\delta_2 = \frac{PL}{AE}$$

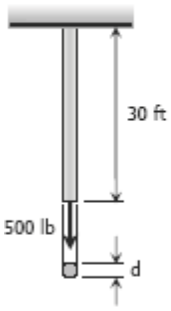
Where: P = 20 kN = 20 000 N
L = 150 m = 150 000 mm
A = 300 mm^2
E = 200 000 MPa

$$\delta_2 = \frac{20000(150000)}{300(200000)} = 50 \text{ mm}$$

Total elongation:
δ = 4.33 + 50 = **54.33 mm**

Example 202 :A steel wire 30 ft long, hanging vertically, supports a load of 500 lb. Neglecting the weight of the wire, determine the required diameter if the stress is not to exceed 20 ksi and the total elongation is not to exceed 0.20 in. Assume E = 29 × 10^6 psi.

Solution 202



Based on maximum allowable stress:

$$\sigma = \frac{P}{A}$$

$$20\,000 = \frac{500}{\frac{1}{4}\pi d^2}$$

$$d = 0.0318 \text{ in}$$

Based on maximum allowable deformation:

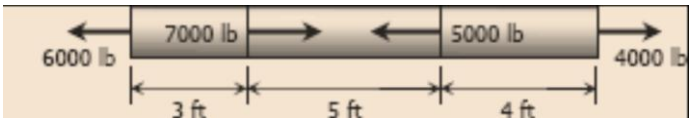
$$\delta = \frac{PL}{AE}$$

$$0.20 = \frac{500(30\times12)}{\frac{1}{4}\pi d^2(29\times10^6)}$$

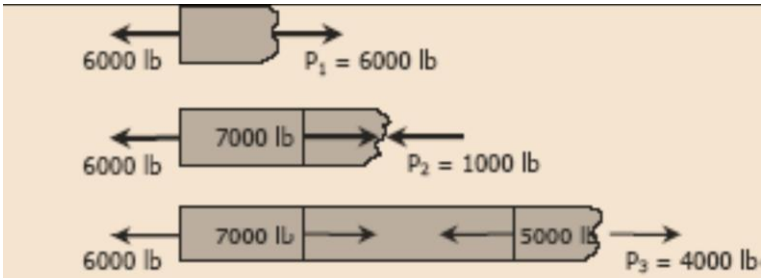
$$d = 0.0395 \text{ in}$$

Use the bigger diameter, d = **0.0395 in**

Example 203 : An aluminum bar having a cross-sectional area of 0.5 in^2 carries the axial loads applied at the positions shown in Fig. P-209. Compute the total change in length of the bar if E = 10 × 10^6 psi. Assume the bar is suitably braced to prevent lateral buckling.



Solution 203



$P_1 = 6000$ lb tension
$P_2 = 1000$ lb compression
$P_3 = 4000$ lb tension

$$\delta = \frac{PL}{AE}$$

$$\delta = \delta_1 - \delta_2 + \delta_3$$

$$\delta = \frac{6000(3\times12)}{0.5(10\times10^6)} - \frac{1000(5\times12)}{0.5(10\times10^6)} + \frac{4000(4\times12)}{0.5(10\times10^6)}$$

$\delta = 0.0696$ in (lengthening)

Example 204: Axial loads are applied at the positions indicated. Find the largest value of P that will not exceed an overall deformation of 3.0 mm, or the following stresses: 140 MPa in the steel, 120 MPa in the bronze, and 80 MPa in the aluminum. Assume that the assembly is suitably braced to prevent buckling. Use E_{st} = 200 GPa, E_{al} = 70 GPa, and E_{br} = 83 GPa.

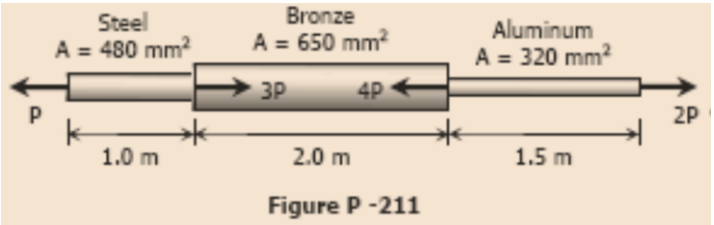


Figure P -211

Solution 204

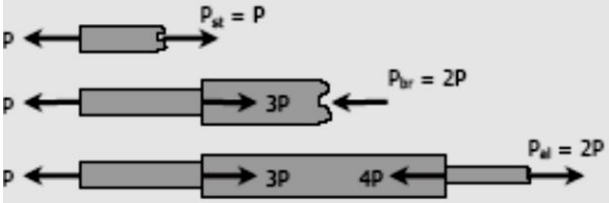
Based on allowable stresses:

Steel:

$P_{st} = \sigma_{st} A_{st}$

$P = 140(480) = 67\,200$ N

$P = 67.2$ kN



Bronze:

$P_{br} = \sigma_{br} A_{br}$

$2P = 120(650) = 78\,000$

$P = 39\,000$ N = 39 kN

Aluminum:

$P_{al} = \sigma_{al} A_{al}$

$2P = 80(320) = 25\,600$ N

$P = 12\,800$ N = 12.8 kN

Based on allowable deformation:
(steel and aluminum lengthens, bronze shortens)

$\delta = \delta_{st} - \delta_{br} + \delta_{al}$

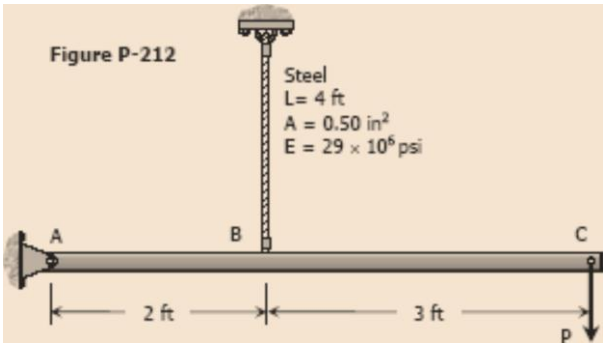
$$3 = \frac{P(1000)}{480(200\,000)} - \frac{2P(2000)}{650(70\,000)} + \frac{2P(1500)}{320(83\,000)}$$

$$3 = \left(\frac{1}{96000} - \frac{1}{11375} + \frac{3}{26560}\right)P$$

$P = 84\,610.99$ N = 84.61 kN

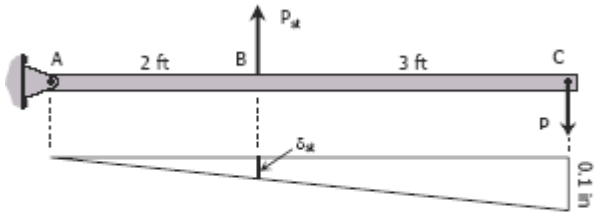
Use the smallest value of P, $P = 12.8$ kN

Example 205 : The rigid bar ABC shown in Fig. P-212 is hinged at A and supported by a steel rod at B. Determine the largest load P that can be applied at C if the stress in the steel rod is limited to 30 ksi and the vertical movement of end C must not exceed 0.10 in.



Solution 205

Free body and deformation diagrams:



Based on maximum stress of steel rod:

$\Sigma M_A = 0$
$5P = 2P_{st}$
$P = 0.4P_{st}$
$P = 0.4\sigma_{st}A_{st}$
$P = 0.4[30(0.50)]$
$P = 6$ kips

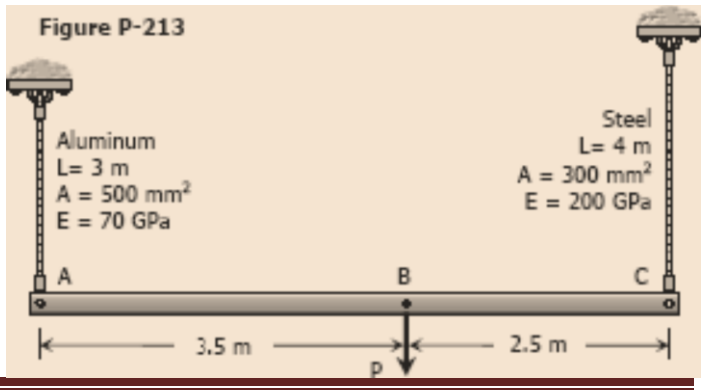
Based on movement at C:

$$\frac{\delta_{st}}{2} = \frac{0.1}{5}$$
$\delta_{st} = 0.04$ in
$$\frac{P_{st}L}{AE} = 0.04$$
$$\frac{P_{st}(4\times12)}{0.50(29\times10^6)} = 0.04$$
$P_{st} = 12\,083.33$ lb

$\Sigma M_A = 0$
$5P - 2P_{st}$
$P = 0.4P_{st}$
$P = 0.4(12\,083.33)$
$P = 4833.33$ lb $= 4.83$ kips

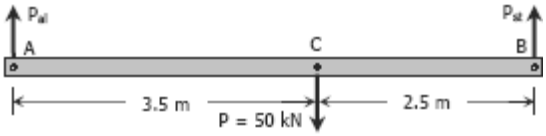
Use the smaller value, $P = 4.83$ kips

Example 206 : The rigid bar AB, attached to two vertical rods as shown in Fig. P-213, is horizontal before the load P is applied. Determine the vertical movement of P if its magnitude is 50 kN.



Solution 206

Free body diagram:



For aluminum:

$[\Sigma M_B = 0]$ $\quad 6P_{al} = 2.5(50)$

$P_{al} = 20.83$ kN

$\left[\delta = \frac{PL}{AE}\right]_{al}$ $\quad \delta_{al} = \frac{20.83(3)1000^2}{500(70000)}$

$\delta_{al} = 1.78$ mm

For steel:

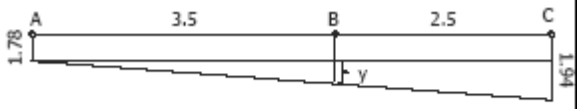
$[\Sigma M_A = 0]$ $\quad 6P_{st} = 3.5(50)$

$P_{st} = 29.17$ kN

$\left[\delta = \frac{PL}{AE}\right]_{st}$ $\quad \delta_{st} = \frac{29.17(4)1000^2}{300(200000)}$

$\delta_{st} = 1.94$ mm

Movement diagram:



$\frac{y}{3.5} = \frac{1.94 - 1.78}{6}$

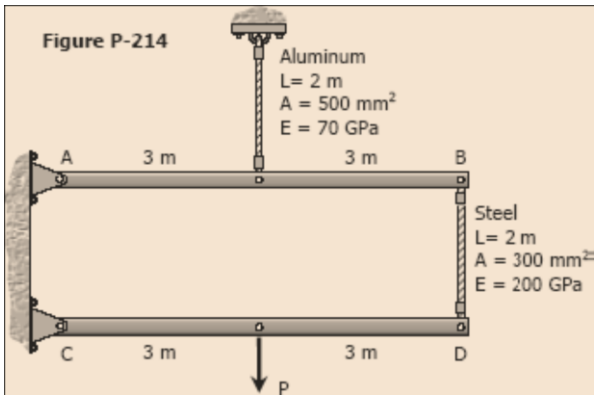
$y = 0.09$ mm

δ_B = vertical movement of P

$\delta_B = 1.78 + y = 1.78 + 0.09$

$\delta_B = 1.87$ mm

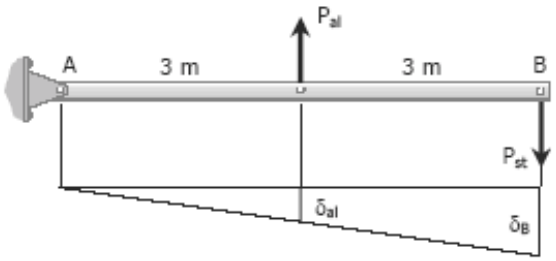
Example 207: The rigid bars AB and CD shown in Fig. P-214 are supported by pins at A and C and the two rods. Determine the maximum force P that can be applied as shown if its vertical movement is limited to 5 mm. Neglect the weights of all members.



Solution 207

$[\sum M_A = 0]$ $\quad 3P_{al} = 6P_{st}$

$P_{al} = 2P_{st}$



FBD and movement diagram of bar AB

By ratio and proportion:

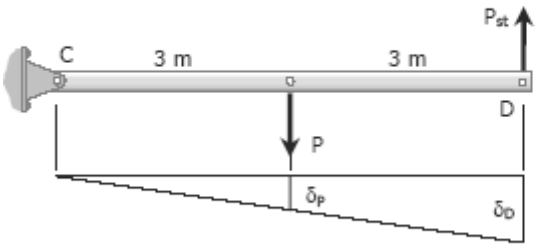
$$\frac{\delta_B}{6} = \frac{\delta_{al}}{3}$$

$$\delta_B = 2\delta_{al} = 2\left[\frac{PL}{AE}\right]_{al}$$

$$\delta_B = 2\left[\frac{P_{al}(2000)}{500(70\,000)}\right]$$

$$\delta_B = \tfrac{1}{8750} P_{al} = \tfrac{1}{8750} (2P_{st})$$

$$\delta_B = \tfrac{1}{4375} P_{st} \rightarrow \text{movement of B}$$



FBD and movement diagram of bar CD

Movement of D:

$$\delta_D = \delta_{st} + \delta_B = \left[\frac{PL}{AE}\right]_{st} + \tfrac{1}{4375} P_{st}$$

$$\delta_D = \frac{P_{st}(2000)}{300(200\,000)} + \tfrac{1}{4375} P_{st}$$

$$\delta_D = \tfrac{11}{42000} P_{st}$$

$[\sum M_C = 0]$ $\quad 6P_{st} = 3P$

$P_{st} = \tfrac{1}{2} P$

By ratio and proportion:

$$\frac{\delta_P}{3} = \frac{\delta_D}{6}$$

$$\delta_P = \tfrac{1}{2}\delta_D = \tfrac{1}{2}\left(\tfrac{11}{42000} P_{st}\right)$$

$$\delta_P = \tfrac{11}{84000} P_{st}$$

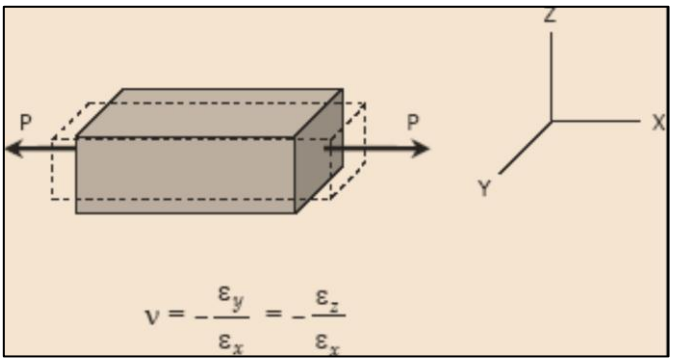
$$5 = \tfrac{11}{84000}\left(\tfrac{1}{2} P\right)$$

$$P = 76\,363.64 \text{ N} = \mathbf{76.4 \text{ kN}}$$

Poisson's Ratio:

If a bar is subjected to a tensile loading there will be an increase in length of the bar in the direction of the applied load, but there is also a decrease in a lateral dimension perpendicular to the load. It has been observed that for an elastic materials, the lateral strain is proportional to the longitudinal strain. The ratio of the lateral strain to longitudinal strain is known as the Poison's ratioand is denoted by ν.

Poison's ratio (ν) = - lateral strain / longitudinal strain



where $\boldsymbol{\varepsilon_x}$ is strain in the x-direction and $\boldsymbol{\varepsilon_y}$ and $\boldsymbol{\varepsilon_z}$ are the strains in the perpendicular direction. The negative sign indicates a decrease in the transverse dimension when $\boldsymbol{\varepsilon_x}$ is positive.

For most engineering materials the value of (ν) is between 0.15 and 0.33.

For most steel, it lies in the range of 0.25 to 0.3, and 0.20 for concrete.

BIAXIAL DEFORMATION:

If an element is subjected simultaneously by Tensile stresses, σ_x and σ_y, in the x and y directions, the strain in the x-direction is σ_x / E and the strain in the y direction is σ_y / E. Simultaneously, the stress in the y direction will produce a lateral contraction on the x-x direction of the amount ($-\nu\,\varepsilon_y$ or $-\nu\,\sigma_y/E$). The resulting strain in the x direction will be :

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \quad \text{or} \quad \sigma_x = \frac{(\varepsilon_x + \nu\varepsilon_y)E}{1-\nu^2}$$

and

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} \quad \text{or} \quad \sigma_y = \frac{(\varepsilon_y + \nu\varepsilon_x)E}{1-\nu^2}$$

TRIAXIAL DEFORMATION

If an element is subjected simultaneously by three mutually perpendicular normal stresses σ_x, σ_y, and σ_z, which are accompanied by strains ε_x, ε_y, and ε_z, respectively,

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

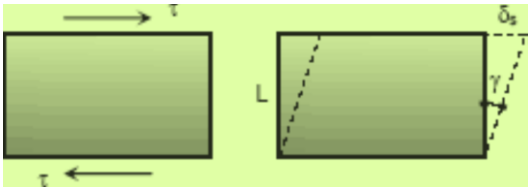
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

Tensile stresses and elongation are taken as positive. Compressive stresses and contraction are taken as negative.

Shear Deformation and Shear Strain

Shearing forces cause shearing deformation. An element subject to shear does not change in length but undergoes a change in shape.



The change in angle at the corner of an original rectangular element is called the **Shear Strain(γ)** and is expressed as:

$$\gamma = \frac{\delta s}{L}$$

The ratio of the shear stress **τ** and the shear strain **γ** is called the **modulus of elasticity** in shear or modulus of rigidity and is denoted as G, in MPa.

$$\mathbf{G} = \frac{\boldsymbol{\tau}}{\boldsymbol{\gamma}}$$

The relationship between the shearing deformation and the applied shearing force is :

$$\delta_s = \frac{VL}{A_s G} = \frac{\tau L}{G}$$

where **V** is the shearing force acting over an area $\mathbf{A_s}$.

Relationship Between E, G, and ν

The relationship between modulus of elasticity E, shear modulus G and Poisson's ratio ν is given as :

$$G = \frac{E}{2(1+\nu)}$$

Bulk Modulus of Elasticity or Modulus of Volume Expansion, K

The bulk modulus of elasticity K is a measure of a resistance of a material to change in volume without change in shape or form. It is given as :

$$K = \frac{E}{3(1-2\nu)} = \frac{\sigma}{\Delta V / V}$$

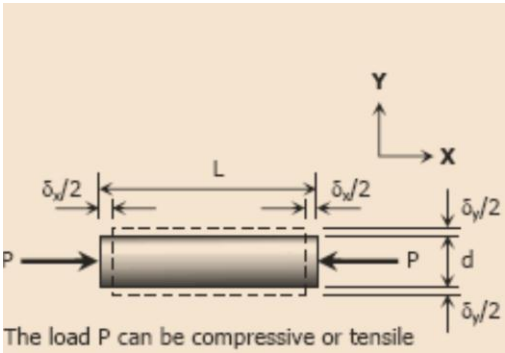
where V is the volume and ΔV is change in volume. The ratio ΔV / V is called **Volumetric Strain** and can be expressed as:

$$\frac{\Delta V}{V} = \frac{\sigma}{K} = \frac{3(1-2\nu)}{E}$$

Solved Problems in Poison's ratio

Problem 222: A solid cylinder of diameter d carries an axial load P. Show that its change in diameter is $4P\nu / \pi Ed$.

Solution 222



$$\nu = -\frac{\varepsilon_y}{\varepsilon_x}$$

$$\varepsilon_y = -\nu\varepsilon_x$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E}$$

$$\frac{\delta_y}{d} = -\nu\frac{-P}{AE}$$

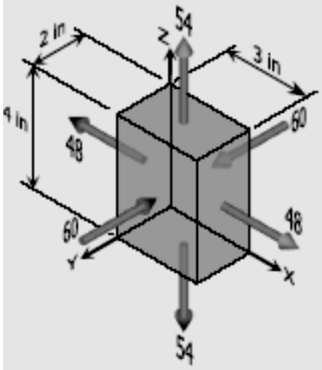
$$\delta_y = \nu\frac{Pd}{\frac{1}{4}\pi d^2 E}$$

$$\delta_y = \frac{4P\nu}{\pi Ed} \quad ok!$$

Problem 223: A rectangular steel block is 3 inches long in the x direction, 2 inches long in the y direction, and 4 inches long in the z direction. The block is subjected to a triaxial loading of three uniformly distributed forces as follows: 48 kips tension in the x direction, 60 kips compression in the y

direction, and 54 kips tension in the z direction. If ν = 0.30 and E = 29×10^6 psi, determine the single uniformly distributed load in the x direction that would produce the same deformation in the y direction as the original loading.

Solution 223



For triaxial deformation (tensile triaxial stresses):
(compressive stresses are negative stresses)

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$\sigma_x = \frac{P_x}{A_{yz}} = \frac{48}{4(2)} = 6.0 \text{ ksi (tension)}$$

$$\sigma_y = \frac{P_y}{A_{xz}} = \frac{60}{4(3)} = 5.0 \text{ ksi (compression)}$$

$$\sigma_z = \frac{P_z}{A_{xy}} = \frac{54}{2(3)} = 9.0 \text{ ksi (tension)}$$

$$\varepsilon_y = \frac{1}{29 \times 10^6}[-5000 - 0.30(6000 + 9000)]$$

$$\varepsilon_y = -3.276 \times 10^{-4}$$

ε_y is negative, thus tensile force is required in the x-direction to produce the same deformation in the y-direction as the original forces.

For equivalent single force in the x-direction:
(uniaxial stress)

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x}$$

$$-\nu\varepsilon_x = \varepsilon_y$$

$$-\nu\frac{\sigma_x}{E} = \varepsilon_y$$

$$-0.30\left(\frac{\sigma_x}{29 \times 10^6}\right) = -3.276 \times 10^{-4}$$

$$\sigma_x = 31\,666.67 \text{ psi}$$

$$\sigma_x = \frac{P_x}{4(2)} = 31\,666.67$$

$$P_x = 253\,333.33 \text{ lb (tension)}$$

$$P_x = \mathbf{253.33 \text{ kips (tension)}}$$

Problem 224 : For the block loaded triaxially as described in Prob. 223, find the uniformly distributed load that must be added in the x-direction to produce no deformation in the z-direction.

Solution 224

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

σ_x = 6.0 ksi (tension)
σ_y = 5.0 ksi (compression)
σ_z = 9.0 ksi (tension)

$$\varepsilon_z = \frac{1}{29\times10^6}[9000 - 0.3(6000 - 5000)]$$

$$\varepsilon_z = 2.07 \times 10^{-5}$$

ε_z is positive, thus positive stress is needed in the x-direction to eliminate deformation in z-direction.

The application of loads is still simultaneous:
(No deformation means zero strain)

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] = 0$$

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

σ_y = 5.0 ksi → (compression)
σ_z = 9.0 ksi → (tension)

$$9000 = 0.30(\sigma_x - 5000)$$

$$\sigma_x = 35\,000 \text{ psi}$$

$$\sigma_{added} + 6000 = 35\,000$$

$$\sigma_{added} = 29\,000 \text{ psi}$$

$$\frac{P_{added}}{2(4)} = 29\,000$$

$$P_{added} = 232\,000 \text{ lb}$$

$$P_{added} = \mathbf{232 \text{ kips}}$$

Problem 225 : A welded steel cylindrical drum made of a 10-mm plate has an internal diameter of 1.20 m. Compute the change in diameter that would be caused by an internal pressure of 1.5 MPa. Assume that Poisson's ratio is 0.30 and E = 200 GPa.

Solution 225

σ_y = longitudinal stress

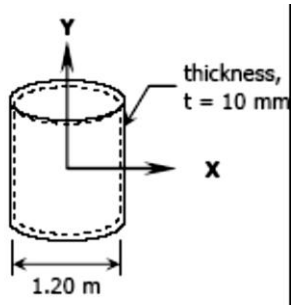
$$\sigma_y = \frac{pD}{4t} = \frac{1.5(1200)}{4(10)}$$

$$\sigma_y = 45 \text{ MPa}$$

σ_x = tangential stress

$$\sigma_x = \frac{pD}{2t} = \frac{1.5(1200)}{2(10)}$$

$$\sigma_x = 90 \text{ MPa}$$



$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$$\varepsilon_x = \frac{90}{200000} - 0.3\left(\frac{45}{200000}\right)$$

$$\varepsilon_x = 3.825 \times 10^{-4}$$

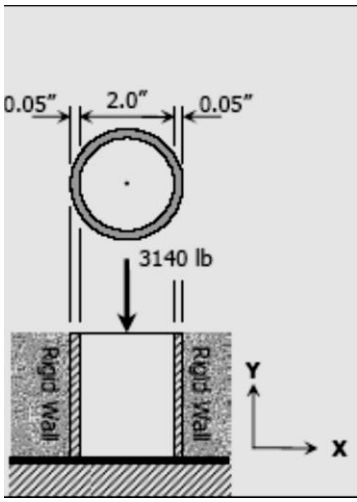
$$\varepsilon_x = \frac{\Delta D}{D}$$

$$\Delta D = \varepsilon_x D = (3.825 \times 10^{-4})(1200)$$

$$\Delta D = \mathbf{0.459 \text{ mm}}$$

Problem 226 : A 2-in.-diameter steel tube with a wall thickness of 0.05 inch just fits in a rigid hole. Find the tangential stress if an axial compressive load of 3140 lb is applied. Assume ν = 0.30 and neglect the possibility of buckling.

Solution 226



$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = 0$$

$$\sigma_x = \nu\sigma_y$$

where σ_x = tangential stress

σ_y = longitudinal stress

$$\sigma_y = \frac{P_y}{A} = \frac{3140}{\pi(2)(0.05)}$$

$$\sigma_y = 31400/\pi \text{ psi}$$

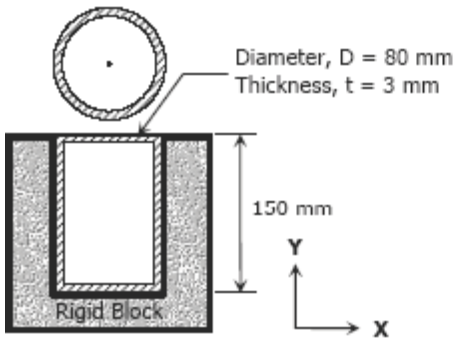
$$\sigma_x = 0.30(31400/\pi)$$

$$\sigma_x = 9430/\pi \text{ psi}$$

$$\sigma_x = 2298.5 \text{ psi}$$

Problem 227 : A 150-mm-long bronze tube, closed at its ends, is 80 mm in diameter and has a wall thickness of 3 mm. It fits without clearance in an 80-mm hole in a rigid block. The tube is then subjected to an internal pressure of 4.00 MPa. Assuming ν = 1/3 and E = 83 GPa, determine the tangential stress in the tube.

Solution 227



Longitudinal stress:

$$\sigma_y = \frac{pD}{4t} = \frac{4(80)}{4(3)}$$

$$\sigma_y = \frac{80}{3} \text{ MPa}$$

The strain in the x-direction is:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = 0$$

$\sigma_x = \nu\sigma_y$ = tangential stress

$$\sigma_x = \frac{1}{3}\left(\frac{80}{3}\right)$$

$$\sigma_x = 8.89 \text{ MPa}$$

CHAPTER 3

Statically Indeterminate Members

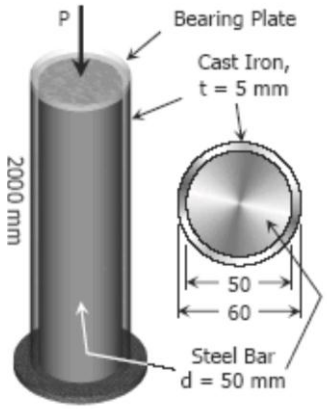
There are many problems, however, in which the internal forces can not be determined from statics alone. In fact, in most of these problems the reactions them selves—which are external forces—can not be determined by simply drawing a free-body diagram of the member and writing the corresponding equilibrium equations. The equilibrium equations must be complemented by relations involving deformations obtained by considering the geometry of the problem.

Because statics is not sufficient to determine either the reactions or the internal forces, problems of this type are said to be **statically indeterminate**. The following examples will show how to handle this type of problems.

Solved Problems in Statically Indeterminate Members:

Problem 201A: Steel bar 50 mm in diameter and 2 m long is surrounded by a shell of a cast iron 5 mm thick. Compute the load that will compress the combined bar a total of 0.8 mm in the length of 2 m. For steel, E = 200 GPa, and for cast iron, E = 100GPa.

Solution:



$$\delta = \frac{PL}{AE}$$

$$\delta = \delta_{cast\ iron} = \delta_{steel} = 0.8\ \text{mm}$$

$$\delta_{cast\ iron} = \frac{P_{cast\ iron}(2000)}{[\frac{1}{4}\pi(60^2 - 50^2)](100\,000)} = 0.8$$

$$P_{cast\ iron} = 11\,000\pi\ \text{N}$$

$$\delta_{steel} = \frac{P_{steel}(2000)}{[\frac{1}{4}\pi(50^2)](200\,000)} = 0.8$$

$$P_{steel} = 50\,000\pi\ \text{N}$$

$$\sum F_V = 0$$

$$P = P_{cast\ iron} + P_{steel}$$

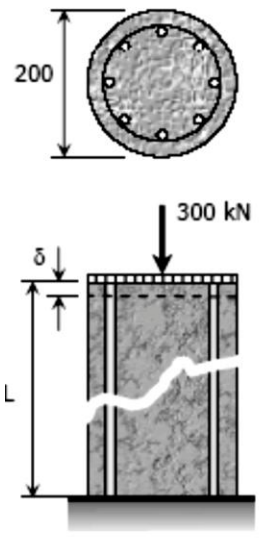
$$P = 11\,000\pi + 50\,000\pi$$

$$P = 61\,000\pi\ \text{N}$$

$$P = \mathbf{191.64\ kN}$$

Problem 202A: Reinforced concrete column 200 mm in diameter is designed to carry an axial compressive load of 300 kN. Determine the required area of the reinforcing steel if the allowable stresses are 6 MPa and 120 MPa for the concrete and steel, respectively. Use E_{co} = 14 GPa and E_{st} = 200 GPa.

Solution 234



$$\delta_{co} = \delta_{st} = \delta$$

$$\left(\frac{PL}{AE}\right)_{co} = \left(\frac{PL}{AE}\right)_{st}$$

$$\left(\frac{\sigma L}{E}\right)_{co} = \left(\frac{\sigma L}{E}\right)_{st}$$

$$\frac{\sigma_{co} L}{14000} = \frac{\sigma_{st} L}{200000}$$

$$100\sigma_{co} = 7\sigma_s$$

When σ_{st} = 120 MPa

$100\sigma_{co} - 7(120)$

σ_{co} = 8.4 MPa > 6 MPa (not ok!)

When σ_{co} = 6 MPa

$100(6) = 7\sigma_{st}$

σ_{st} = 85.71 MPa < 120 MPa (ok!)

Use σ_{co} = 6 MPa and σ_{st} = 85.71 MPa

$$\sum F_V = 0$$

$$P_{st} + P_{co} = 300$$

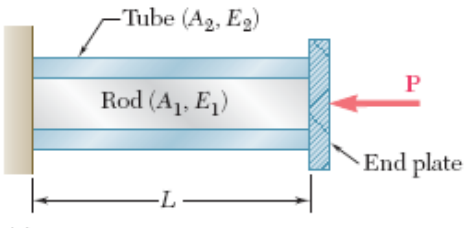
$$\sigma_{st} A_{st} + \sigma_{co} A_{co} = 300$$

$$85.71A_{st} + 6[\tfrac{1}{4}\pi(200)^2 - A_{st}] = 300(1000)$$

$$79.71A_{st} + 60\,000\pi = 300\,000$$

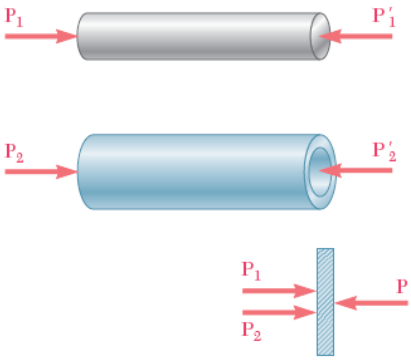
$$A_{st} = \mathbf{1398.9\ mm^2}$$

Problem 203: A rod of length *L*, cross-sectional area A_1, and modulus of elasticity E_1, has been placed inside a tube of the same length *L*, but of cross-sectional area A_2 and modulus of elasticity E_2. What is the deformation of the rod and tube when a force **P** is exerted on a rigid end plate as shown?



Solution:

Denoting by *P1* and *P2*, respectively, the axial forces in the rod and in the tube, we draw free-body diagrams of all three elements. Only the last of the diagrams yields any significant information, namely:



$P_1+P_2=P$ ------(1)

Clearly, one equation is not sufficient to determine the two unknown internal forces P_1 and P_2. The problem is statically indeterminate.

However, the geometry of the problem shows that the deformations

$\boldsymbol{\delta}1$ and $\boldsymbol{\delta}2$ of the rod and tube must be equal. We can write :

$$\boldsymbol{\delta 1} = \frac{\boldsymbol{P1L}}{\boldsymbol{A1E1}} \text{ and } \boldsymbol{\delta 2} = \frac{\boldsymbol{P2L}}{\boldsymbol{A2E2}} \quad \text{----------(2)}$$

Equating the deformations $\boldsymbol{\delta}1$ and $\boldsymbol{\delta}2$, we obtain

$$\frac{\boldsymbol{P1}}{\boldsymbol{A1E1}} = \frac{\boldsymbol{P2}}{\boldsymbol{A2E2}} \quad \text{---------- (3)}$$

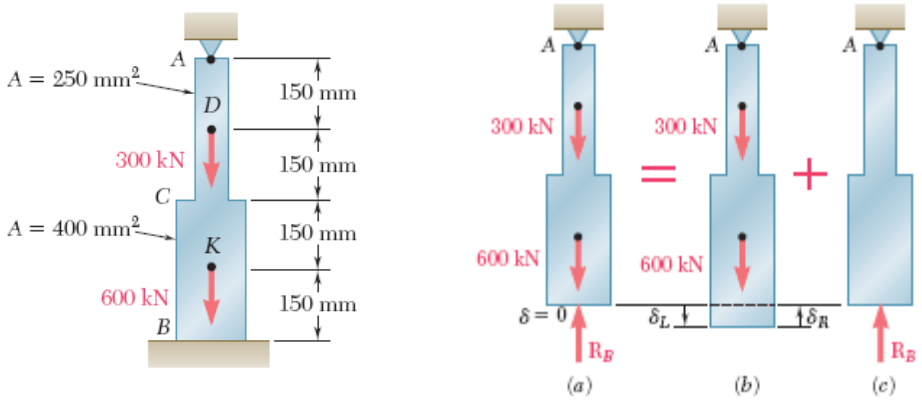
Equations (1) and (3) can be solved simultaneously for P_1 and P_2:

$$P_1 = \frac{A_1E_1P}{A_1E_1 + A_2E_2} \qquad P_2 = \frac{A_2E_2P}{A_1E_1 + A_2E_2}$$

Either of Eqs. (2) can then be used to determine the common deformation of the rod and

Superposition Method. We observe that a structure is statically indeterminate when ever it is held by more supports than are required to maintain its equilibrium. This results in more unknown reactions than available equilibrium equations. It is often found convenient to designate one of the reactions as ***redundant*** and to eliminate the corresponding support. Since the stated conditions of the problem cannot be arbitrarily changed, the ***redundant*** reaction must be maintained in the solution. But it will be treated as an ***unknown load*** that, together with the other loads, must produce deformations that are compatible with the original constraints. The actual solution of the problem is carried out by considering separately the deformations caused by the given loads and by the redundant reaction, and by adding—or *superposing*—the results obtained.

Problem 204: Determine the reactions at *A* and *B* for the steel bar and loading shown in Fig. 2.24, assuming a close fit at both supports before the loads are applied.



Solution:

We consider the reaction at *B* as redundant and release the bar from that support. The reaction R_B is now considered as an unknown load (*a*) and will be determined from the condition that the deformation d of the rod must be equal to zero. The solution is carried out by considering separately the deformation $\boldsymbol{\delta_L}$ caused by the given loads (*b*) and the

deformation δ_R due to the redundant reaction $R_B(c)$.

The deformation δ_L is obtained from Eq. (2.8) after the bar has been divided into four portions, as shown in Fig. 2.26. Following the same procedure as in Example 2.01, we write

$$P_1 = 0 \qquad P_2 = P_3 = 600 \times 10^3\ \text{N} \qquad P_4 = 900 \times 10^3\ \text{N}$$
$$A_1 = A_2 = 400 \times 10^{-6}\ \text{m}^2 \qquad A_3 = A_4 = 250 \times 10^{-6}\ \text{m}^2$$
$$L_1 = L_2 = L_3 = L_4 = 0.150\ \text{m}$$

Substituting these values into Eq. (2.8), we obtain

$$\delta_L = \sum_{i=1}^{4} \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3\ \text{N}}{400 \times 10^{-6}\ \text{m}^2} + \frac{600 \times 10^3\ \text{N}}{250 \times 10^{-6}\ \text{m}^2} + \frac{900 \times 10^3\ \text{N}}{250 \times 10^{-6}\ \text{m}^2}\right)\frac{0.150\ \text{m}}{E}$$
$$\delta_L = \frac{1.125 \times 10^9}{E} \qquad (2.17)$$

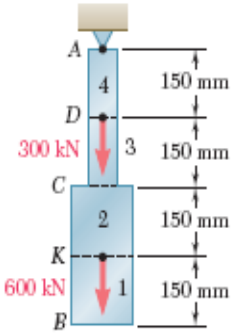


Fig. 2.26

Considering now the deformation δ_R due to the redundant reaction R_B, we divide the bar into two portions, as shown in Fig. 2.27, and write

$$P_1 = P_2 = -R_B$$
$$A_1 = 400 \times 10^{-6}\ \text{m}^2 \qquad A_2 = 250 \times 10^{-6}\ \text{m}^2$$
$$L_1 = L_2 = 0.300\ \text{m}$$

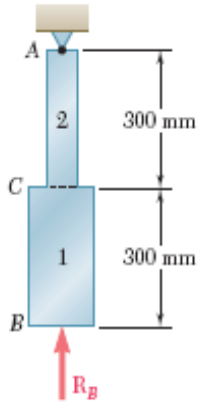


Fig. 2.27

Substituting these values into Eq. (2.8), we obtain

$$\delta_R = \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E} = -\frac{(1.95 \times 10^3) R_B}{E} \qquad (2.18)$$

Expressing that the total deformation δ of the bar must be zero, we write

$$\delta = \delta_L + \delta_R = 0 \qquad (2.19)$$

and, substituting for δ_L and δ_B from (2.17) and (2.18) into (2.19),

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3) R_B}{E} = 0$$

Solving for R_B, we have

$$R_B = 577 \times 10^3\ \text{N} = 577\ \text{kN}$$

The reaction R_A at the upper support is obtained from the free-body diagram of the bar (Fig. 2.28). We write

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300\ \text{kN} - 600\ \text{kN} + R_B = 0$$
$$R_A = 900\ \text{kN} - R_B = 900\ \text{kN} - 577\ \text{kN} = 323\ \text{kN}$$

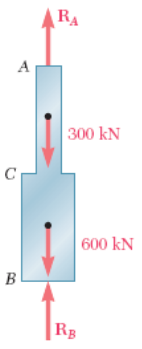
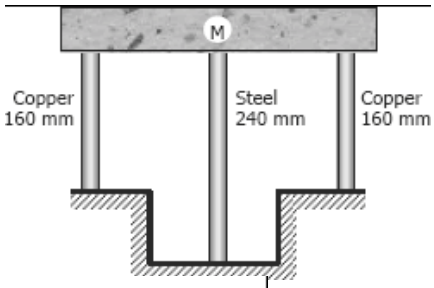
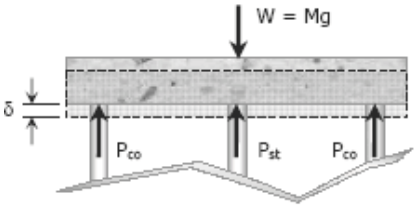


Fig. 2.28

Once the reactions have been determined, the stresses and strains in the bar can easily be obtained. It should be noted that, while the total deformation of the bar is zero, each of its component parts *does deform* under the given loading and restraining conditions.

Problem 205: A rigid block of mass M is supported by three symmetrically spaced rods as shown in Figure. Each copper rod has an area of 900 mm^2; E = 120 GPa; and the allowable stress is 70 MPa. The steel rod has an area of 1200 mm^2; E = 200 GPa; and the allowable stress is 140 MPa. Determine the largest mass M which can be supported.





$\delta_{co} = \delta_{st}$

$$\left(\frac{\sigma L}{E}\right)_{co} = \left(\frac{\sigma L}{E}\right)_{st}$$

$$\frac{\sigma_{co}(160)}{120000} = \frac{\sigma_{st}(240)}{200000}$$

$$10\sigma_{co} = 9\sigma_{st}$$

When σ_{st} = 140 MPa

$\sigma_{co} = \frac{9}{10}(140)$

σ_{co} = 126 MPa > 70 MPa (not ok!)

When σ_{co} = 70 MPa

$\sigma_{st} = \frac{10}{9}(70)$

σ_{st} = 77.78 MPa < 140 MPa (ok!)

Use σ_{co} = 70 MPa and σ_{st} = 77.78 MPa

$\sum F_V = 0$

$2P_{co} + P_{st} = W$

$2(\sigma_{co}A_{co}) + \sigma_{st}A_{st} = Mg$

$2[70(900)] + 77.78(1200) = M(9.81)$

M = **22 358.4 kg**

H.W

In Prob. 205, How should the lengths of the two identical copper rods be changed so that each material will be stressed to its allowable limit?

CHAPTER 4

Thermal Stresses

Temperature changes cause the body to expand or contract. The amount $\boldsymbol{\delta_T}$, is given by :

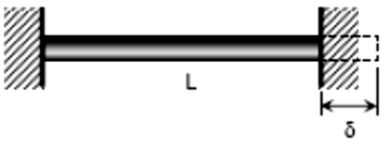
$$\delta_T = \alpha L(T_f - T_i) = \alpha L\, \Delta T$$

where ($\boldsymbol{\alpha}$) is the coefficient of thermal expansion in m/m°C, L is the length in meter, and (T_i and T_f) are the initial and final temperatures, respectively in °C.

For steel, $\boldsymbol{\alpha = 11.25 \times 10^{-6} / C^{\circ}}$.

stress will be induced in the structure. In some cases where temperature deformation is not permitted, an internal stress is created. The internal stress created is termed as thermal stress.

For a homogeneous rod mounted between unyielding supports as shown, the thermal stress is computed as:



deformation due to temperature changes;

$$\delta_T = \alpha L\, \Delta T$$

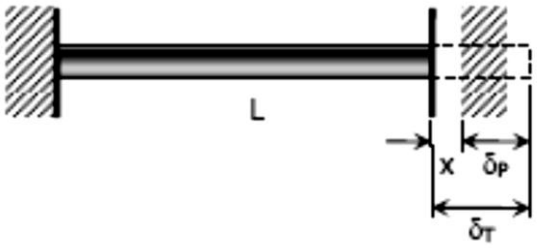
deformation due to equivalent axial stress;

$$\delta_P = \frac{PL}{AE} = \frac{\sigma L}{E}$$
$$\delta_T = \delta_P$$
$$\alpha L\,\Delta T = \frac{\sigma L}{E}$$
$$\sigma = E\,\alpha\,\Delta T$$

where (**σ**) is the thermal stress in MPa and E is the modulus of elasticity of the rod in MPa.

If the wall yields a distance of (x) as shown, the following calculations will be made:

$$\delta_T = x + \delta_P$$
$$\alpha L\,\Delta T = x\,\frac{\sigma L}{E}$$



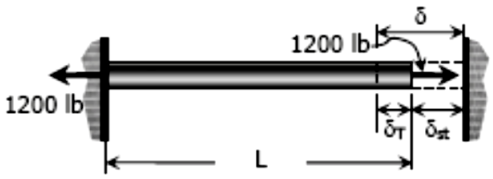
where (**σ**) represents the thermal stress.

Take note that as the temperature rises above the normal, the rod will be in compression, and if the temperature drops below the normal, the rod is in tension.

Solved Problems in Thermal Stress

Problem 261: A steel rod with a cross-sectional area of 0.25 in^2 is stretched between two fixed points. The tensile load at 70°F is 1200 lb. What will be the stress at 0°F? At what temperature will the stress be zero? Assume $\boldsymbol{\alpha}$ = 6.5 × 10^{-6} in / (in·°F) and E = 29 × 10^6 psi.

Solution 261



For the stress at 0°C:

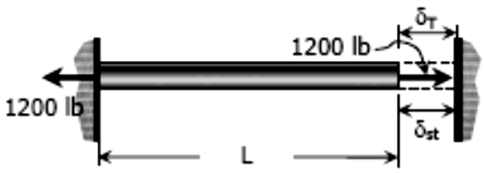
$$\delta = \delta_T + \delta_{st}$$

$$\frac{\sigma L}{E} = \alpha L(\Delta T) + \frac{PL}{AE}$$

$$\sigma = \alpha E(\Delta T) + \frac{P}{A}$$

$$\sigma = (6.5 \times 10^{-6})(29 \times 10^{6})(70) + \frac{1200}{0.25}$$

$$\sigma = 17\,995 \text{ psi} = \mathbf{18 \text{ ksi}}$$



For the temperature that causes zero stress:

$$\delta_T = \delta_{st}$$

$$\alpha L(\Delta T) = \frac{PL}{AE}$$

$$(6.5 \times 10^{-6})(T - 70) = \frac{1200}{0.25(29 \times 10^{6})}$$

$$T = \mathbf{95.46°C}$$

Problem 262: A steel rod is stretched between two rigid walls and carries a tensile load of 5000 N at 20°C. If the allowable stress is not to exceed 130 MPa at -20°C, what is the minimum diameter of the rod? Assume **α** = 11.7 μm/(m·°C) and E = 200 GPa.

Solution 262

$$\delta = \delta_T + \delta_{st}$$

$$\frac{\sigma L}{E} = \alpha L(\Delta T) + \frac{PL}{AE}$$

$$\sigma = \alpha E(\Delta T) + \frac{P}{A}$$

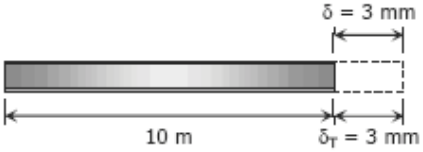
$$130 = (11.7 \times 10^{-6})(200\,000)(40) + \frac{5000}{A}$$

$$A = \frac{5000}{36.4} = 137.36 \text{ mm}^2$$

$$\tfrac{1}{4}\pi d^2 = 137.36; \quad d = 13.22 \text{ mm}$$

Problem 263: Steel railroad reels 10 m long are laid with a clearance of 3 mm at a temperature of 15°C. At what temperature will the rails just touch? What stress would be induced in the rails at that temperature if there were no initial clearance? Assume **α** = 11.7 μm/(m·°C) and E = 200 GPa.

Solution 263



Temperature at which $\delta_T = 3$ mm:

$$\delta_T = \alpha L(\Delta T)$$

$$\delta_T = \alpha L(T_f - T_i)$$

$$3 = (11.7 \times 10^{-6})(10\,000)(T_f - 15)$$

$$T_f = \mathbf{40.64°C}$$

Required stress:

$$\delta = \delta_T$$

$$\frac{\sigma L}{E} = \alpha L(\Delta T)$$

$$\sigma = \alpha E(T_f - T_i)$$

$$\sigma = (11.7 \times 10^{-6})(200\,000)(40.64 - 15)$$

$$\sigma = \mathbf{60 \text{ MPa}}$$

Problem 264: A steel rod 3 feet long with a cross-sectional area of 0.25 in.^{2} is stretched between two fixed points. The tensile force is 1200 lb at 40°F. Using $E = 29 \times 10^6$ psi and $\alpha = 6.5 \times 10^{-6}$ in./(in.·°F), calculate (a) the temperature at which the stress in the bar will be 10 ksi; and (b) the temperature at which the stress will be zero.

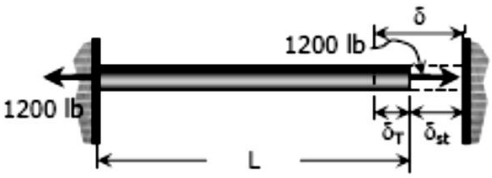
Solution 264

(a) Without temperature change:

$\sigma = P/A = 1200/0.25 = 4800$ psi

$\sigma = 4.8$ ksi < 10 ksi

A drop of temperature is needed to increase the stress to 10 ksi. See accompanying figure.



$$\delta = \delta_T + \delta_{st}$$

$$\frac{\sigma L}{E} = \alpha L(\Delta T) + \frac{PL}{AE}$$

$$\sigma = \alpha E(\Delta T) + \frac{P}{A}$$

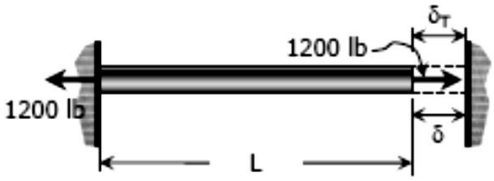
$$10\,000 = (6.5 \times 10^{-6})(29 \times 10^6)(\Delta T) + \frac{1200}{0.25}$$

$$\Delta T = 27.59°\text{F}$$

Required temperature:
(temperature must drop from 40°F)

$T = 40 - 27.59 = \mathbf{12.41°F}$

(b) From the figure below:



$$\delta = \delta_T$$

$$\frac{PL}{AE} = \alpha L(\Delta T)$$

$$P = \alpha AE(T_f - T_i)$$

$$1200 = (6.5 \times 10^{-6})(0.25)\,(29 \times 10^6)(T_f - 40)$$

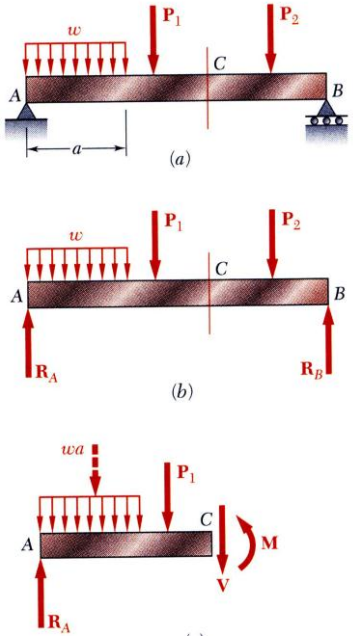
$$T_f = \mathbf{65.46°F}$$

CHAPTER 5

BEAMS

Introduction :

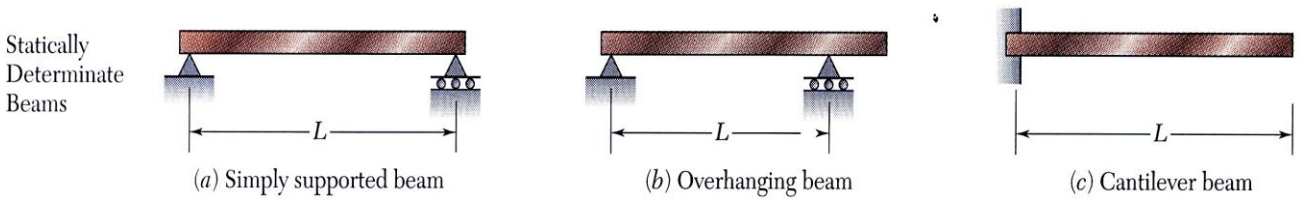
- *Beams* - structural members supporting loads at various points along the member.
- Transverse loadings of beams are classified as *concentrated* loads or *distributed* loads.
- Applied loads result in internal forces consisting of a shear force (from the shear stress distribution) and a bending couple (from the normal stress distribution).



Classification of Beams:

1- Statically Determinate Beams:

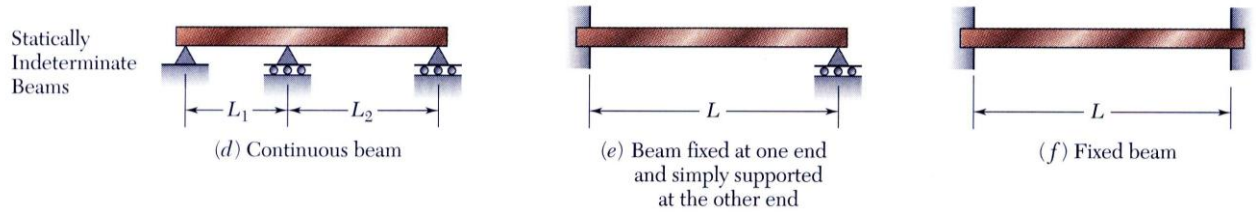
Statically determinate beams are those beams in which the reactions of the supports may be determined by the use of the equations of static equilibrium. The beams shown below are examples of statically determinate beams.



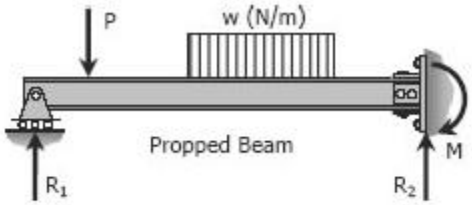
(a) Simply supported beam (b) Overhanging beam (c) Cantilever beam

2- Statically Indeterminate Beams:

If the number of reactions exerted upon a beam exceeds the number of equations in static equilibrium, the beam is said to be statically indeterminate. In order to solve the reactions of the beam, the static equations must be supplemented by equations based upon the elastic deformations of the beam.

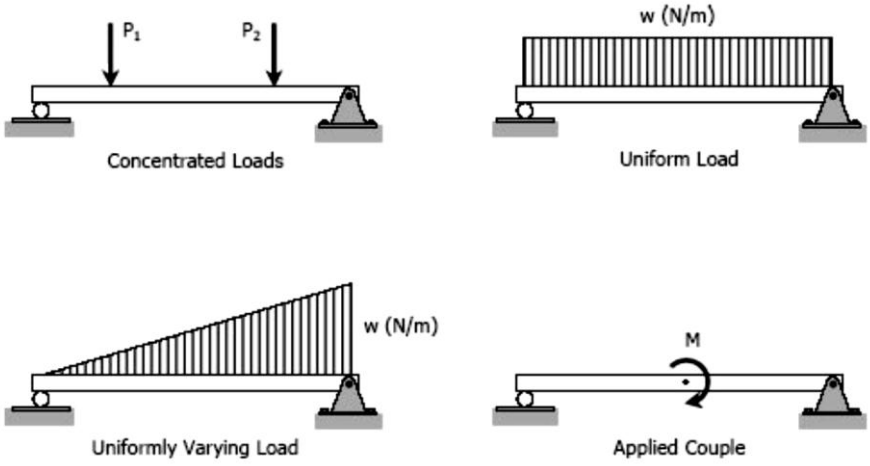


The degree of indeterminacy is taken as the difference between the number of reactions to the number of equations in static equilibrium that can be applied. In the case of the propped beam shown, there are three reactions (R_1, R_2, and M) while only two equations ($\Sigma M = 0$ and $\Sigma Fv = 0$) can be applied, thus the beam is indeterminate to the first degree (3 – 2 = 1).



TYPES OF LOADING

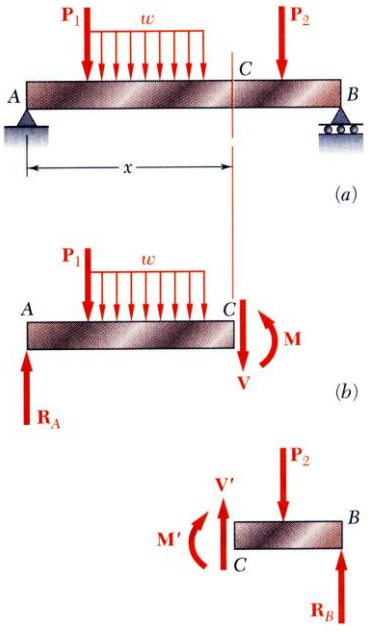
Loads applied to the beam may consist of a concentrated load (load applied at a point), uniform load, uniformly varying load, or an applied couple or moment. These loads are shown in the following figures.



Shear Force and Bending Moment Diagrams

Shear Force and Bending Moment Diagrams are plots of the shear forces and bending moments, respectively, along the length of a beam. The purpose of these plots is to clearly show maximum of the shear force and bending moment, which are important in the design of beams.

The most common sign convention for the shear force and bending moment in beams is shown in Fig. 9.12.



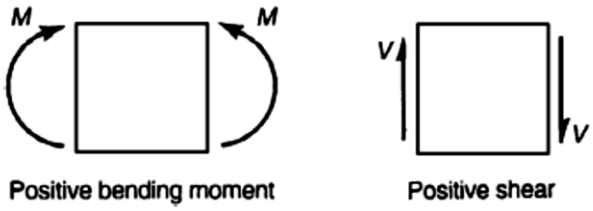


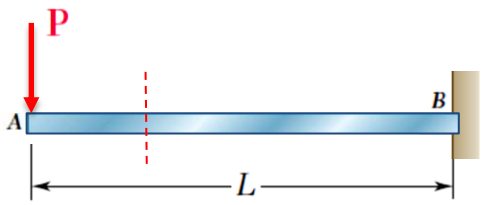
Figure 9.12 Sign convention for bending moment and shear

One method of determining the shear and moment diagrams is by the following steps:

1. Determine the reactions from equilibrium of the entire beam.
2. Cut the beam at an arbitrary point.
3. Show the unknown shear and moment on the cut using the positive sign convention shown in Fig. 9.12.
4. Sum forces in the vertical direction to determine the unknown shear.
5. Sum moments about the cut to determine the unknown moment.

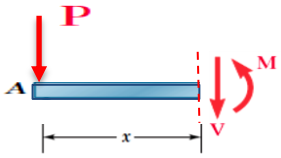
Example (1)

For the beam shown, derive equations for shear force and bending moment at any point along the beam.



Solution:

We cut the beam at a point between A and B at distance x from A and draw thefree-body diagram of the left part of the beam, directing **V** and **M** as indicated in the figure.



$\Sigma F_y = 0$:

$$P + V = 0$$

$$V = -P \quad (\downarrow)$$

$\Sigma M_x = 0$:

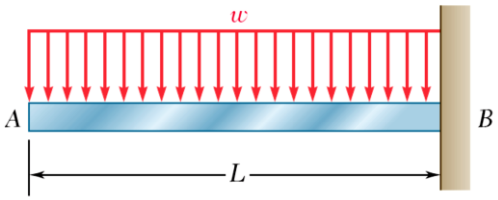
$$P.x + M = 0$$

$$M = - Px(\quad) \quad \curvearrowleft$$

- ***Note*** *that shear force is constant (equal P) along the beam, and bending moment is a linear function of* ***(x)****.*

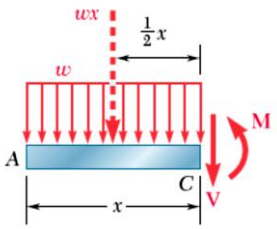
Example (2):

For a cantilever beam AB of span L supporting a uniformly distributed load w, derive equations for shear force and bending moment at any point along the beam.



Solution:

We cut the beam at a point *C* between *A* and *B* and draw free-body diagram of *AC*, directing **V** and **M** as indicated in Fig. Denoting by *x* the distance from *A* to *C* and replacing the distributed load over *AC* by its resultant (*wx*) applied at the mid point of *AC*, we write:



$\Sigma F_y = 0$:

$-wx - V = 0$

$V = -wx$

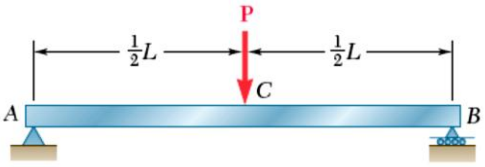
$\Sigma M_x = 0$:

$$wx\left(\frac{x}{2}\right) + M = 0$$

$$M = -\frac{1}{2}wx^2$$

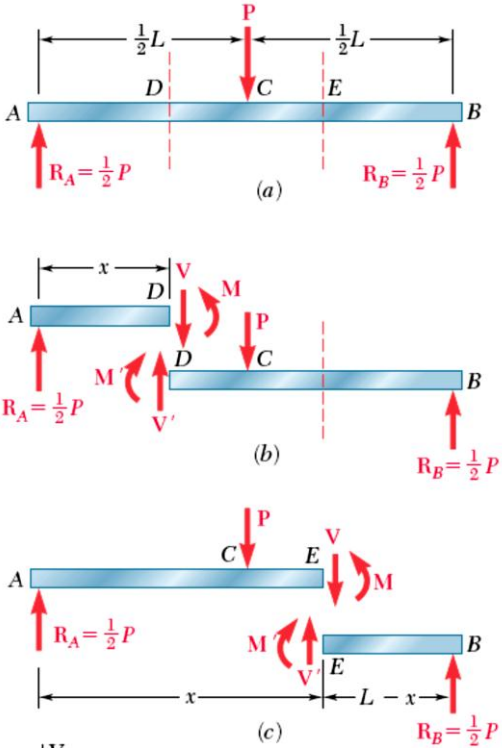
Example (3):

For the simply supported beam ***AB*** of span ***L*** supporting a single concentrated load ***P***, derive equations for shear force and bending moment at any point along the beam.



Solution:

We first determine the reactions at the supports from the free-body diagram of the entire beam; we find that the magnitude of each reaction is equal to $P/2$. Next we cut the beam at a point D between A and C and draw the free-body diagrams of AD and DB. *Assuming that shear and bending moment are positive,* we direct the internal forces **V** and **V'** and the internal couples **M** and **M'** as indicated in Fig. Considering the free body AD and writing that the sum of the vertical components and the sum of the moments about D of the forces acting on the free body are zero, we find:



$$V = +P/2 \quad \text{and} \quad M = +Px/2.$$

Both the shear and bending moment are therefore positive; this may be checked by observing that the reaction at A tends to shear off and to bend the beam at D as indicated in Figs. *b* and *c*. The shear has a constant value $V = P/2$, while the bending moment increases linearly from $M = 0$ at $x = 0$ to $M = PL/4$ at $x = L/2$.

Cutting, now, the beam at a point E between C and B and considering the free body EB (Fig. *c*), we write that the sum of the vertical components and the sum of the moments about E of the forces acting on the free body are zero. We obtain:

$$V = -P/2 \text{ and } M = P(L - x)/2.$$

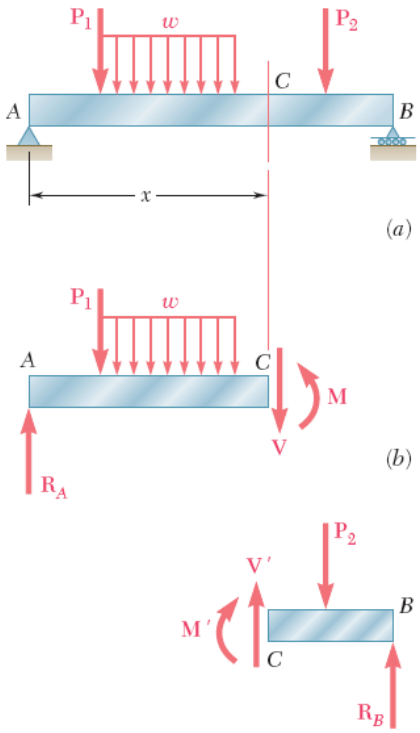
The shear is therefore negative and the bending moment positive; this can be checked by observing that the reaction at B bends the beam at E as indicated in Fig. *c* but tends to shear it off in a manner opposite to that shown in Fig. *b*.

Note *that the shear has a constant value $V = -P/2$ between C and B, while the bending moment decreases linearly from $M = PL/4$ at $x = L/2$ to $M = 0$ at $x = L$.*

Shear Force and Moment Diagram

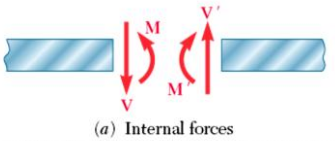
The determination of the maximum absolute values of the shear and of the bending moment in a beam are greatly facilitated if *V* and *M* are plotted against the distance *x* measured from one end of the beam. Besides, as you will see later, the knowledge of *M* as a function of *x* is essential to the determination of the deflection of a beam.

In the examples and sample problems of this section, the shear and bending-moment diagrams will be obtained by determining the values of *V* and *M* at selected points of the beam. These values will be found in the usual way, i.e., by passing a section through the point where they are to be determined (Fig. *a*) and considering the equilibrium of the portion of beam located on either side of the section (Fig. *b*). Since the shear forces **V** and **V** have opposite senses, recording the shear at point *C* with an up or down arrow would be meaning less, unless we indicated at the same time which of the free bodies *AC* and *CB* we are considering.



For this reason, the shear *V* will be recorded with a sign: a *plus sign* if the shearing forces are directed as shown in Fig.*a*, and a *minus sign* otherwise. A similar convention will apply for the bending moment *M*. It will be considered as positive if the bending couples are directed as shown in that figure, and negative otherwise. Summarizing the sign conventions we have presented, we state:

The shear V and the bending moment M at a given point of abeam are said to be positive when the internal forces and couples actingon each portion of the beam are directed as shown in Fig. a.



(*a*) Internal forces
(positive shear and positive bending moment)

Example (4): For the beam shown, plot the shear and moment diagram.

Solution:

First**,** solve for the unknown reactions using the free-body diagram of the beam shown in Fig, (a). to find the reactions, sum moments about the left end which gives:

$$6R_2 - (3)(2) = 0 \text{ or } R_2 = 6/6 = 1 \text{ kN}$$

Sum forces in the vertical direction to get:

$$R_1 + R_2 = 3 = R_1 + 1 \text{ or } R_1 = 2 \text{ kN}$$

Cut the beam between the left end and the load as shown in (b). Show the unknown moment and shear on the cut using the positive sign convention. Sum the vertical forces to get:

$V = 2$ kN (independent of x)

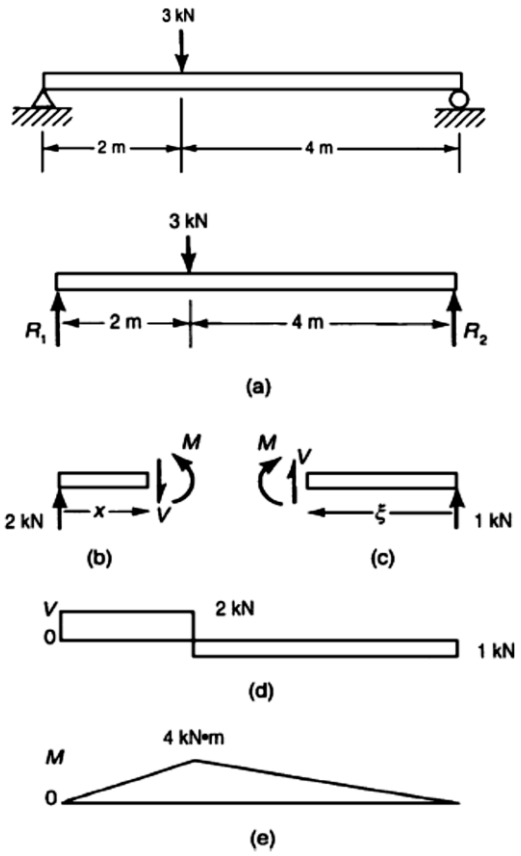
Sum moments about the cut to get:

$M = R_1 x = 2x$

Repeat the procedure by making a cut between the right end of the beam and the 3-kN load, as shown in (c). Again, sum vertical forces and sum moments about the cut to get:

$V = 1$ kN (independent of x), and $M = 1x$

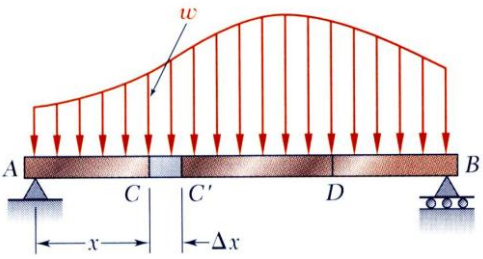
The plots of these expressions for shear and moment give the shear and moment diagrams (as shown in Fig.(d) and (e).

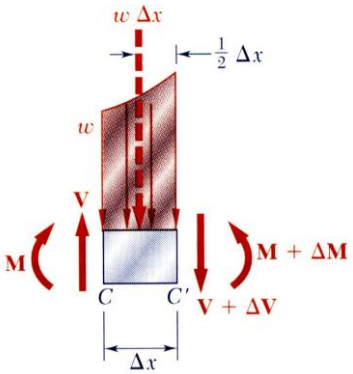


• It should be noted that the shear diagram in this example has a jump at the point of the load and that the jump is equal to the load. This is always the case. Similarly, a moment diagram will have a jump equal to an applied concentrated moment. In this example, there was no concentrated moment applied, so the moment was everywhere continuous.

• Another useful way of determining the shear and moment diagram is by using differential relationships. These relationships are found by considering an element of length Δx of the beam.

The forces on that element are shown in Fig.





$$\sum F_y = 0: \quad V - (V + \Delta V) - w\Delta x = 0$$

$$\Delta V = -w\Delta x$$

Summation of forces in the *y* direction gives :

which gives:

$$\frac{dV}{dx} = -w$$

$$V_D - V_C = -\int_{x_C}^{x_D} w\, dx$$

$$\sum M_{C'} = 0: \quad (M + \Delta M) - M - V\Delta x + w\Delta x \frac{\Delta x}{2} = 0$$

$$\Delta M = V\Delta x - \frac{1}{2} w(\Delta x)^2$$

$$\frac{dM}{dx} = V$$

$$M_D - M_C = \int_{x_C}^{x_D} V\, dx$$

The load $q = -w_0$, so Eq. (9.59) reads

$$V = V_0 - \int_0^x w_0\,dx = \frac{w_0 L}{2} - w_0 x$$

Noting that the moment at $x = 0$ is zero, Eq. (9.60) gives

$$M = M_0 - \int_0^x \left(\frac{w_0 L}{2} - w_0 x\right) dx = 0 + \frac{w_0 L x}{2} - \frac{w_0 x^2}{2} = \frac{w_0 x}{2}(L - x)$$

Example(5) The simply supported uniform beam shown in Exhibit 16 carries a uniform load of $\mathbf{w_0}$. Plot the shear and moment diagrams for this beam.

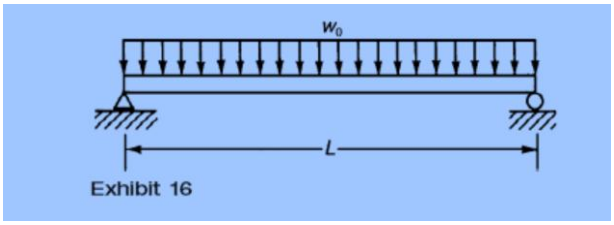


Exhibit 16

Solution

As before, the reactions can be found first from the free-body diagram of the beam shown in Exhibit 17(a). It can be seen that, from symmetry, $R_1 = R_2$. Summing vertical forces then gives

$$R = R_1 = R_2 = \frac{w_0 L}{2}$$

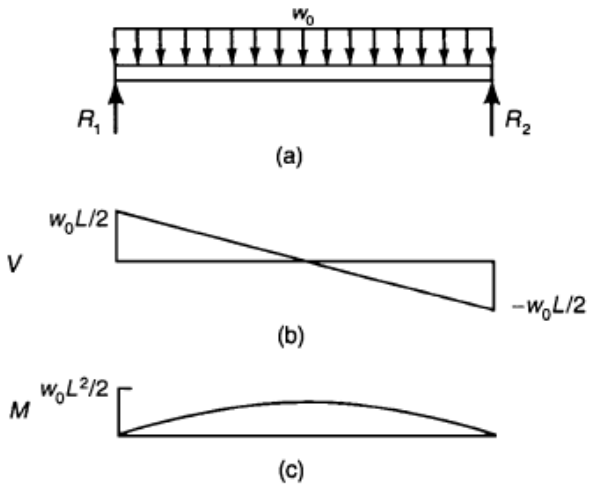


Exhibit 17

It can be seen that the shear diagram is a straight line, and the moment varies parabolically with *x*. Shear and moment diagrams are shown in Exhibit 17(b) and Exhibit 17(c). It can be seen that the maximum bending moment occurs at the center of the beam where the shear stress is zero. The maximum bending moment always has a relative maximum at the place where the shear is zero because the shear is the derivative of the moment, and relative maxima occur when the derivative is zero.

Solved problems

Write shear and moment equations for the beams in the following problems. In each problem, let x be the distance measured from left end of the beam. Also, draw shear and moment diagrams, specifying values at all change of loading positions and at points of zero shear. Neglect the mass of the beam in each problem.

Problem 403

Beam loaded as shown in Fig. P-403.

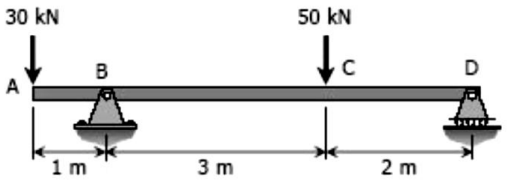


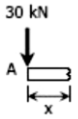
Figure P-403

Solution:

From the load diagram:
$\sum M_B = 0$
$5R_D + 1(30) = 3(50)$
$R_D = 24$ kN

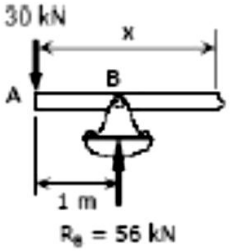
$\sum M_D = 0$
$5R_B = 2(50) + 6(30)$
$R_B = 56$ kN

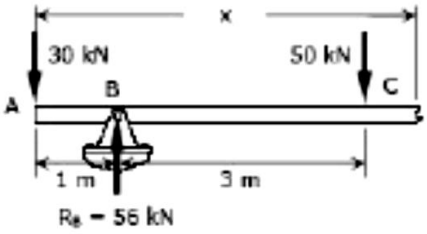
Segment *AB*:
$V_{AB} = -30$ kN
$M_{AB} = -30x$ kN·m



Segment BC:

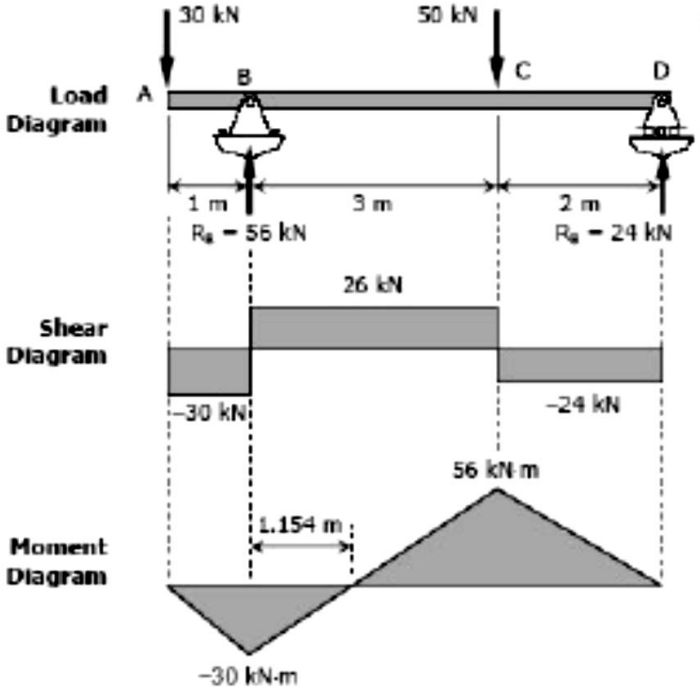
$V_{BC} = -30 + 56$
$= 26$ kN
$M_{BC} = -30x + 56(x - 1)$
$= 26x - 56$ kN·m





Segment CD:

$V_{CD} = -30 + 56 - 50$
$= -24$ kN
$M_{CD} = -30x + 56(x - 1) - 50(x - 4)$
$= -30x + 56x - 56 - 50x + 200$
$= -24x + 144$



To draw the Shear Diagram:

(1) In segment AB, the shear is uniformly distributed over the segment at a magnitude of −30 kN.

(2) In segment BC, the shear is uniformly distributed at a magnitude of 26 kN.

(3) In segment CD, the shear is uniformly distributed at a magnitude of −24 kN.

To draw the Moment Diagram:

(1) The equation $M_{AB} = -30x$ is linear, at $x = 0$, $M_{AB} = 0$ and at $x = 1$ m, $M_{AB} = -30$ kN·m.

(2) $M_{BC} = 26x - 56$ is also linear. At $x = 1$ m, $M_{BC} = -30$ kN·m; at $x = 4$ m, $M_{BC} = 48$ kN·m. When $M_{BC} = 0$, $x = 2.154$ m, thus the moment is zero at 1.154 m from B.

(3) $M_{CD} = -24x + 144$ is again linear. At $x = 4$ m, $M_{CD} = 48$ kN·m; at $x = 6$ m, $M_{CD} = 0$.

CHAPTER 6

STRESSES IN BEAMS

Forces and couples acting on the beam cause bending (flexural stresses) and shearing stresses on any cross section of the beam and deflection perpendicular to the longitudinal axis of the beam. If couples are applied to the ends of the beam and no forces act on it, the bending is said to be pure bending. If forces produce the bending, the bending is called ordinary bending.

Flexure Formula:

Assumptions

1) A plane section of the beam normal to its longitudinal axis prior to loading remains plane after the forces and couples have been applied.
2) The beam is initially straight and of uniform cross section.
3) The moduli of elasticity in tension and compression are equal.
4) The stresses and strains are small (within elastic range), material is homogeneous and Hooks law is applied.

Deformations In A Symmetric Member in Pure Bending

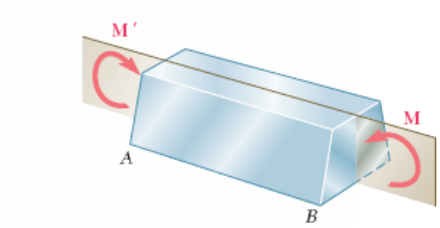


Fig.1 Member in pure bending.

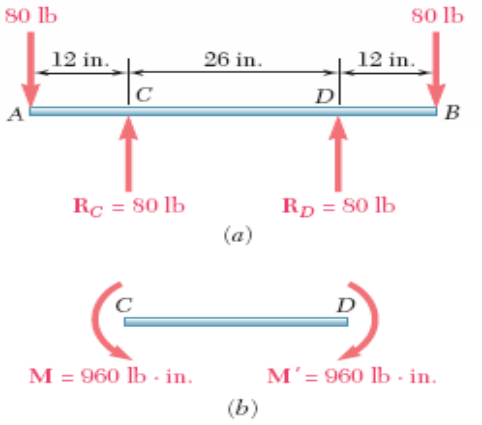


Fig. 2 Beam in which portion *CD* is in pure bending.

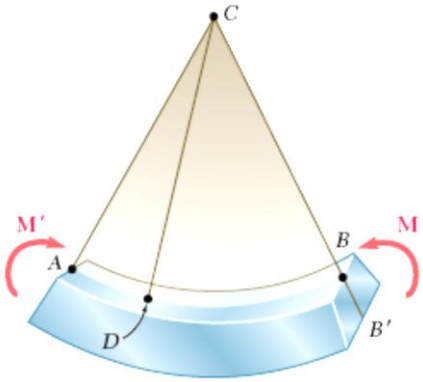
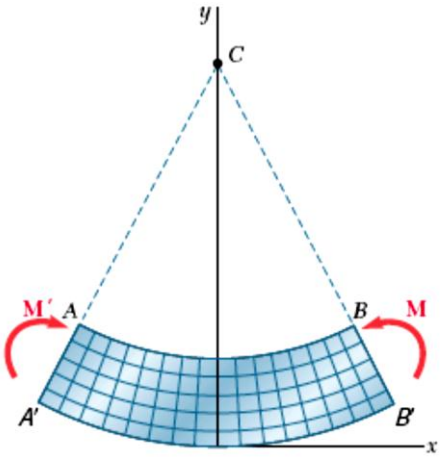
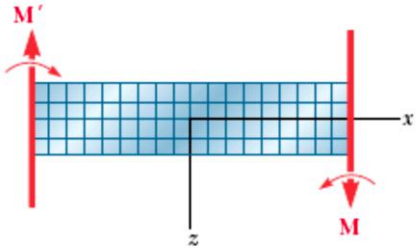


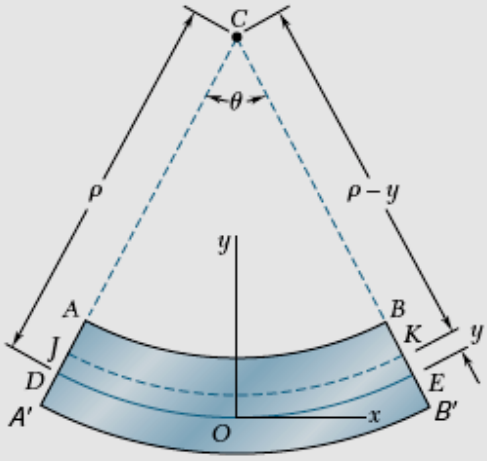
Fig.3 Deformation of member in pure bending.



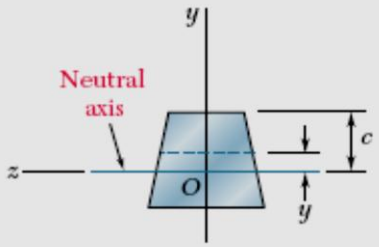
(a) Longitudinal, vertical section (plane of symmetry)



(b) Longitudinal, horizontal section



(a) Longitudinal, vertical section (plane of symmetry)



(b) Transverse section

Fig. 4 Deformation with Respect to Neutral Axis

Denoting by ρ the radius of arc *DE* (Fig. 4-*a*), by Θ the central angle corresponding to *DE*, and observing that the length of *DE* is equal to the length *L* of the unreformed member, we write :

$$L = \rho\theta \qquad \text{----------(1)}$$

The arc *JK* located at a distance *y* above the neutral surface, we note that its length is :

$$L' = (\rho - y)\theta \qquad \text{----------(2)}$$

Since the original length of arc *JK* was equal to *L*, the deformation of *JK* is:

$$\delta = L' - L \qquad \text{----------(3)}$$

or, if we substitute from (1) and (2) into (3), :

$$\delta = (\rho - y)\theta - \rho\theta = -y\theta \qquad \text{----------(4)}$$

The longitudinal strain ϵ_x in the elements of *JK* is obtained by dividing δ by the original length *L* of *JK*. We write:

$$\epsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta}$$

Or

$$\epsilon_x = -\frac{y}{\rho} \qquad \text{----------(5)}$$

Because of the requirement that transverse sections remain plane, identical deformations will occur in all planes parallel to the plane of symmetry. Thus the value of the strain given by Eq. (5) is valid anywhere, and we conclude that the ***longitudinal normal strain ε_x varies linearly with the distance y from the neutral surface.***

The strain ε_x reaches its maximum absolute value when *y* itself is largest. Denoting by ***c*** the largest distance from the neutral surface (which corresponds to either the upper or the lower surface of the member), and by ε_m the *maximum absolute value* of the strain, we have:

$$\epsilon_m = \frac{c}{\rho} \quad \text{---------- (6)}$$

Solving (6) for **ρ** and substituting the value obtained into (5), we can also write:

$$\epsilon_x = -\frac{y}{c}\epsilon_m \quad \text{----------(7)}$$

Stresses And Deformations In The Elastic Range

We now consider the case when the bending moment *M* is such that the normal stresses in the member remain below the yield strength σ_y. This means that, for all practical purposes, the stresses in the member will remain below the proportional limit and the elastic limit as well. There will be no permanent deformation, and Hooke's law for uniaxial stress applies. Assuming the material to be homogeneous, and denoting by ***E*** its modulus of elasticity, we have in the longitudinal ***x*** direction:

$$\sigma_x = E\epsilon_x \quad \text{----------(8)}$$

Recalling Eq. (7), and multiplying both members of that equation by *E*, we write:

$$E\epsilon_x = -\frac{y}{c}(E\epsilon_m)$$

Or

$$\sigma_x = -\frac{y}{c}\sigma_m \quad \text{----------(9)}$$

where σ_m denotes the *maximum absolute value* of the stress. This result shows that, *in the elastic range, the normal stress varies linearly with the distance from the neutral surface* (Fig.5).

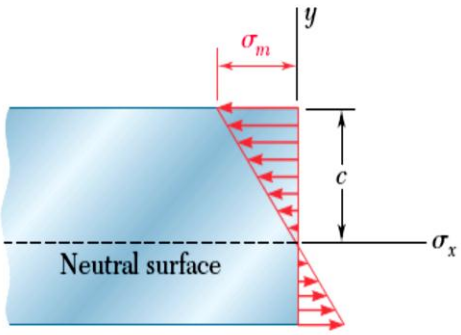


Fig. 5 Bending stresses.

It should be noted that, at this point, we do not know the location of the neutral surface, nor the maximum value σ_m of the stress. Both can be found if we recall the equations of equilibrium which were obtained earlier from statics.
Substituting first for σ_m from (5) into

$$\sum F_x = 0 , \qquad \int \sigma_x \, dA = 0$$

we write :

$$\int \sigma_x \, dA = \int \left(-\frac{y}{c}\sigma_m \right) dA = -\frac{\sigma_m}{c} \int y \, dA = 0$$

from which it follows that:

$$\int y \, dA = 0 \qquad \text{----------(10)}$$

This equation shows that the first moment of the cross section about its neutral axis must be zero. In other words, for a member subjected to pure bending, and ***as long as the stresses remain in the elastic range, the neutral axis passes through the centroid of the section.***

We now recall the 3rd. Eq. of equilibrium, with respect to an *arbitrary* horizontal **z** - axis,

$$\sum M_z = 0 \quad , \qquad \int (-y\sigma_x \, dA) = M$$

Specifying that the **z** - axis should coincide with the neutral axis of the cross section, we substitute for σ_x from (9) and write :

$$\int (-y) \left(-\frac{y}{c}\sigma_m \right) dA = M$$

$$\frac{\sigma_m}{c}\int y^2 dA = M$$

But $$\int y^2 dA = I$$

(*where* I *is the second moment, of the cross section with respect to a centroidal axis perpendicular to the plane of the couple* M), then we can write :

$$\sigma_m = \frac{Mc}{I} \quad \text{---------- (11)}$$

And ,

$$\sigma_x = -\frac{My}{I} \quad \text{---------- (12)}$$

Equations (11) and (12) are called the ***elastic flexure formulas***, and the normal stress σ_x caused by the bending or "flexing" of the member is often referred to as the ***flexural stress***. We verify that the stress is compressive ($\sigma_x > 0$) above the neutral axis ($y > 0$) when the bending moment M is positive, and tensile ($\sigma_x < 0$) when M is negative.

Note:: from now, in this chapter the notation (f_b will be used instead of σ_x to denote the flexural stress.

Now we can write:

$$f_b = \frac{My}{I} \quad \text{and} \quad (f_b)_{max} = \frac{Mc}{I}$$

SECTION MODULUS

In the formula, $(f_b)_{max} = \frac{Mc}{I} = \frac{M}{I/c}$

the ratio ***I/c*** is called the ***Section Modulus*** and is usually denoted by ***S*** with units of mm^3 or (in^3). The maximum bending stress may then be written as :

$$(f_b)_{max} = \frac{M}{S}$$

This form is convenient because the values of ***S*** are available in handbooks for a wide range of standard structural shapes.

The deformation of the member caused by the bending moment M is measured by the curvature of the neutral surface. The curvature (***k***) is defined as the reciprocal of the radius of curvature ρ, and can be obtained by from :

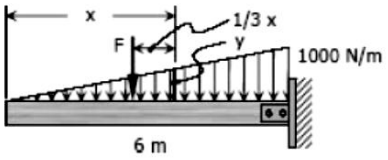
Then , $\frac{1}{\rho} = \frac{\epsilon_m}{c}$ $\frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$

$$\frac{1}{\rho} = \frac{M}{EI} \quad \text{---------- (13)}$$

SOLVED PROBLEMS IN FLEXURE FORMULA

Problem 503 A: cantilever beam, 50 mm wide by 150 mm high and 6 m long, carries a load that varies uniformly from zero at the free end to 1000 N/m at the wall. (a) Compute the magnitude and location of the maximum flexural stress. (b) Determine the type and magnitude of the stress in a fiber 20 mm from the top of the beam at a section 2 m from the free end.

Solution 503



$$M = F(\tfrac{1}{3}x)$$

$$\frac{y}{x} = \frac{1000}{6}$$

$$y = \tfrac{500}{3}x$$

$$F = \tfrac{1}{2}xy$$

$$F = \tfrac{1}{2}x(\tfrac{500}{3}x)$$

$$F = \tfrac{250}{3}x^2$$

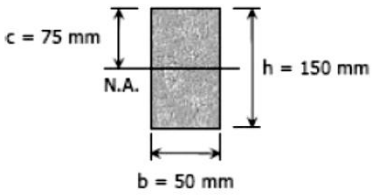
thus $$M = \tfrac{250}{3}x^2(\tfrac{1}{3}x)$$

$$M = \tfrac{250}{9}x^3$$

(a) The maximum moment occurs at the support (the wall) or at $x = 6$ m.

$$M = \tfrac{250}{9}x^3 = \tfrac{250}{9}(6^3)$$

$$= 6000 \text{ N·m}$$

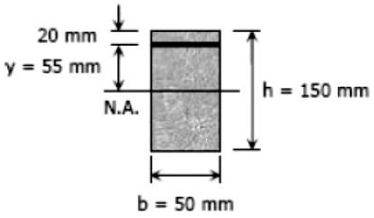


$$(f_b)_{max} = \frac{Mc}{I} = \frac{Mc}{\frac{bh^3}{12}}$$

$$(f_b)_{max} = \frac{6000(1000)(75)}{\frac{50(150)^3}{12}}$$

$$(f_b)_{max} = \mathbf{32\ MPa}$$

(b) At a section 2 m from the free end or at $x = 2$ m at fiber 20 mm from the top of the beam:



$$M = \tfrac{250}{9}x^3 = \tfrac{250}{9}(2)^3$$

$$M = \tfrac{2000}{9} \text{ N·m}$$

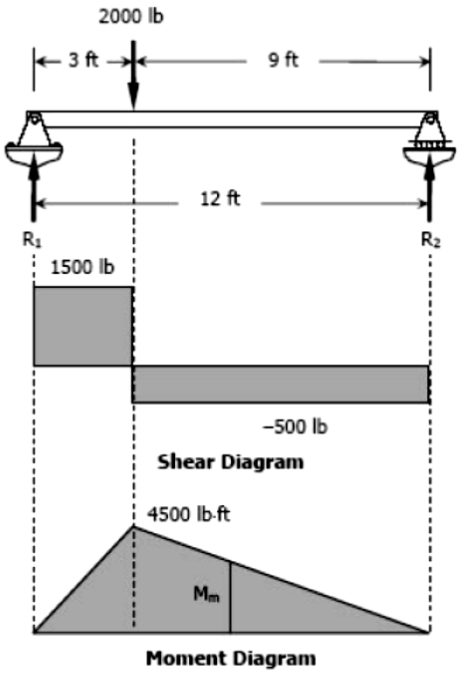
$$f_b = \frac{My}{I} = \frac{(\frac{2000}{9})(1000)(55)}{\frac{50(150)^3}{12}}$$

$$f_b = 0.8691 \text{ MPa} = \mathbf{869.1\ kPa}$$

Problem 504

A simply supported beam, 2 in wide by 4 in high and 12 ft long is subjected to a concentrated load of 2000 lb at a point 3 ft from one of the supports. Determine the maximum fiber stress and the stress in a fiber located 0.5 in from the top of the beam at midspan.

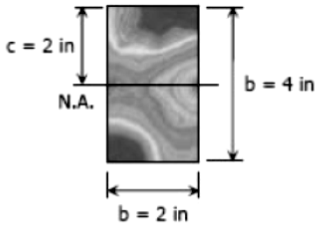
Solution 504



$\Sigma M_{R2} = 0$
$12R_1 = 9(2000)$
$R_1 = 1500$ lb

$\Sigma M_{R1} = 0$
$12R_2 = 3(2000)$
$R_2 = 500$ lb

Maximum fiber stress:



$$(f_b)_{max} = \frac{Mc}{I} = \frac{4500(12)(2)}{\frac{2(4)^3}{12}}$$

$(f_b)_{max} = \mathbf{10{,}125\ psi}$

Stress in a fiber located 0.5 in from the top of the beam at midspan:

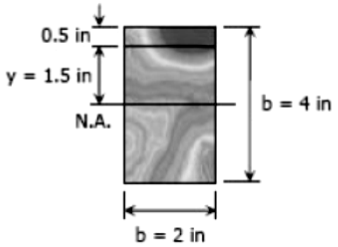
$$\frac{M_m}{6} = \frac{4500}{9}$$

$M_m = 3000$ lb·ft

$$f_b = \frac{My}{I}$$

$$f_b = \frac{3000(12)(1.5)}{\frac{2(4^3)}{12}}$$

$f_b = \mathbf{5{,}062.5\ psi}$

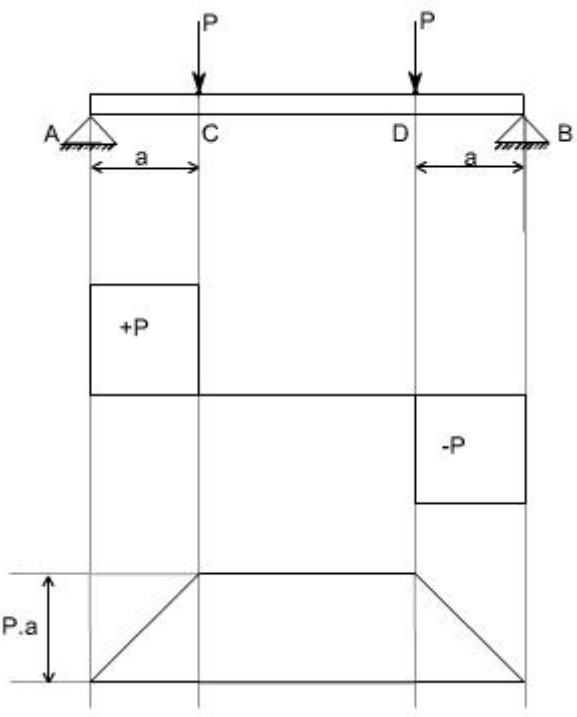


Shearing Stresses in Beams

All the theory which has been discussed earlier, while we discussed the bending stresses in beams was for the case of pure bending i.e. constant bending moment acts along the entire length of the beam.

Let us consider the beam AB transversely loaded as shown in the figure above. Together with shear force and bending moment diagrams we note that the middle potion CD of the beam is free from shear force and that its bending moment. M = P.a is uniform between the portion C and D. This condition is called the pure bending condition.

Since shear force and bending moment are related to each other F= dM/dX (eq) therefore if the shear force changes than there will be a change in the bending moment also, and then this won't be the pure bending.



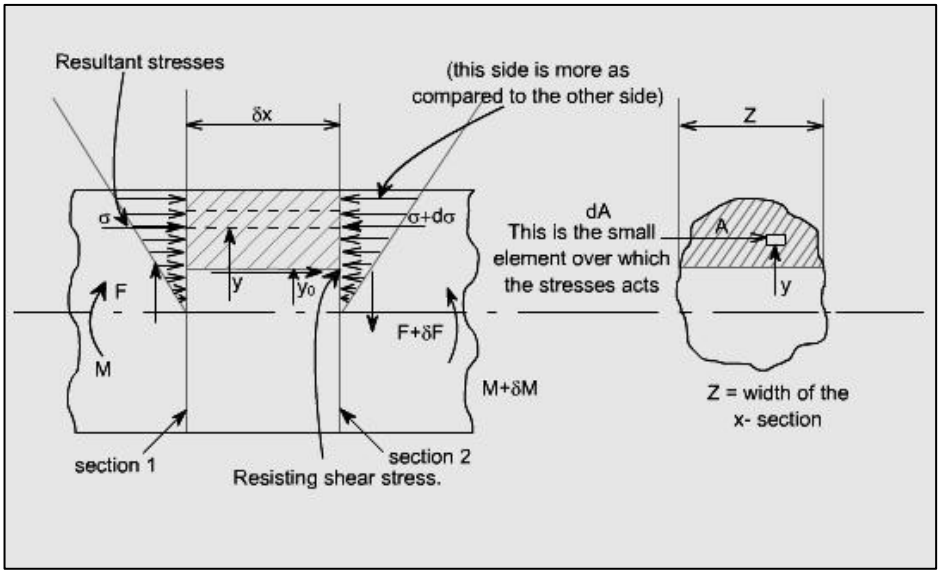
Conclusions: Hence one can conclude from the pure bending theory was that the shear force at each X-section is zero and the normal stresses due to bending are the only ones produced.

Let us study the shear stresses in the beams.

Concept of Shear Stresses in Beams :

By the earlier discussion we have seen that the bending moment represents the resultant of certain linear distribution of normal stresses $\boldsymbol{\delta}_x$ over the cross-section. Similarly, the shear force F_x over any cross-section must be the resultant of a certain distribution of shear stresses.

Derivation of equation for shearing stress :



Assumptions :

1. ***Stress is uniform across the width (i.e. parallel to the neutral axis)***
2. ***The presence of the shear stress does not affect the distribution of normal bending stresses.***

It may be noted that the assumption no.2 cannot be rigidly true as the existence of shear stress will cause a distortion of transverse planes, which will no longer remain plane.

In the above figure let us consider the two transverse sections which are at a distance **' dx'** apart. The shearing forces and bending moments being **F, F + dF** and **M, M + dM** respectively. Now due to the shear stress on transverse planes there will be a complementary shear stress on longitudinal planes parallel to the neutral axis.

Let $\boldsymbol{\tau}$ be the value of the complementary shear stress (and hence the transverse shear stress) at a distance y_o from the neutral axis. **Z** is the width of the x-section at this position

A is area of cross-section cut-off by a line parallel to the neutral axis.

$\bar{y}$ = distance of the centroid of area from the neutral axis.

Let $\boldsymbol{\delta}$, $\boldsymbol{\delta}$+ $\mathbf{d}\boldsymbol{\delta}$ are the normal stresses on an element of area **dA** at the two transverse sections, then there is a difference of longitudinal forces equal to ($\mathbf{d}\boldsymbol{\delta}$ **. dA)** , and this quantity summed over the area **A** is in equilibrium with the transverse shear stress **τ** on the longitudinal plane of area **zdx** .

i.e $\tau.z\delta x = \int d\sigma.dA$

from the bending theory equation

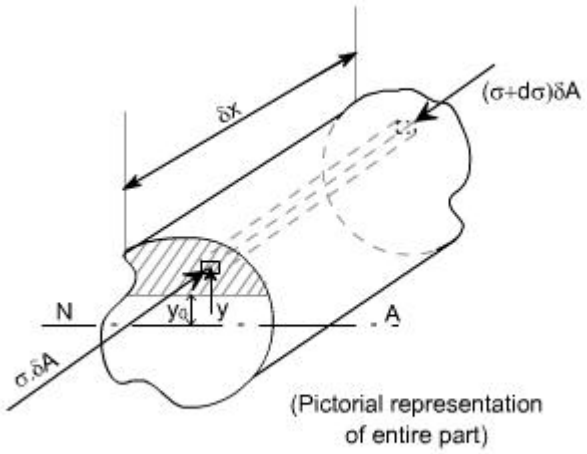
$$\frac{\sigma}{y} = \frac{M}{I}$$

$$\sigma = \frac{M \,.\, y}{I}$$

$$\sigma + d\sigma = \frac{(M + \delta M) \,.\, y}{I}$$

Thus $$d\sigma = \frac{\delta M.y}{I}$$

The figure shown below indicates the pictorial representation of the part.



(Pictorial representation of entire part)

$$d\sigma = \frac{\delta M.y}{I}$$

$$\tau.z\delta x = \int d\sigma.dA$$

$$= \int \frac{\delta M \,.\, y.\delta A}{I}$$

$$\tau.z\delta x = \frac{\delta M}{I} \int y.\delta A$$

But $$F = \frac{\delta M}{\delta x}$$

i.e. $$\tau = \frac{F}{I.z} \int y.\delta A$$

But from definition, $\int y.dA = A\bar{y}$

$\int y.dA$ is the first moment of area of the shaded portion

and $\bar{y}$ = centroid of the area'A'

Hence

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

Where ‘z' is the actual width of the section at the position where ‘ τ ' is being calculated and I is the total moment of inertia about the neutral axis.

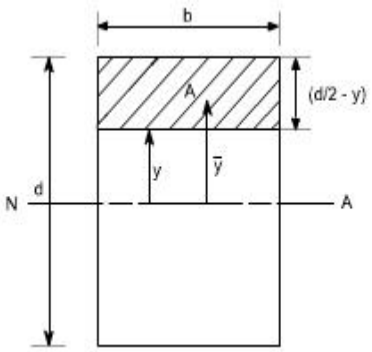
Shearing stress distribution in typical cross-sections:

Let us consider few examples to determine the sheer stress distribution in a given X-sections.

Rectangular x-section:

Consider a rectangular x-section of dimension **b** and **d** .

A is the area of the x-section cut off by a line parallel to the neutral axis. $\bar{y}$ is the distance of the centroid of **A** from the neutral axis.



$$\tau = \frac{F.A.\bar{y}}{I.z}$$

for this case, $A = b\left(\frac{d}{2} - y\right)$

While $\bar{y} = \left[\frac{1}{2}\left(\frac{d}{2} - y\right) + y\right]$

i.e $\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$ and $z = b; I = \frac{b.d^3}{12}$

substituting all these values, in the formula

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$= \frac{F.b.\left(\frac{d}{2} - y\right).\frac{1}{2}.\left(\frac{d}{2} + y\right)}{b.\frac{b.d^3}{12}}$$

$$= \frac{\frac{F}{2}.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{\frac{b.d^3}{12}}$$

$$= \frac{6.F.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{b.d^3}$$

This shows that there is a parabolic distribution of shear stress with y.

The maximum value of shear stress would obviously beat the location y = 0.

Such that $\tau_{max} = \frac{6.F}{b.d^3} \cdot \frac{d^2}{4}$

$= \frac{3.F}{2.b.d}$

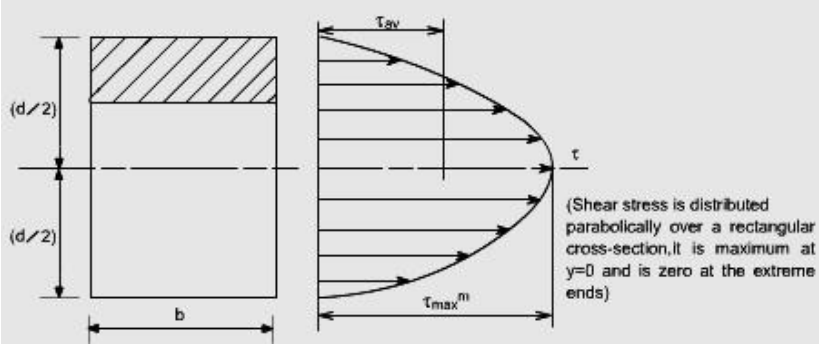
So $\boxed{\tau_{max} = \frac{3.F}{2.b.d}}$ The value of τ_{max} occurs at the neutral axis

The mean shear stress in the beam is defined as

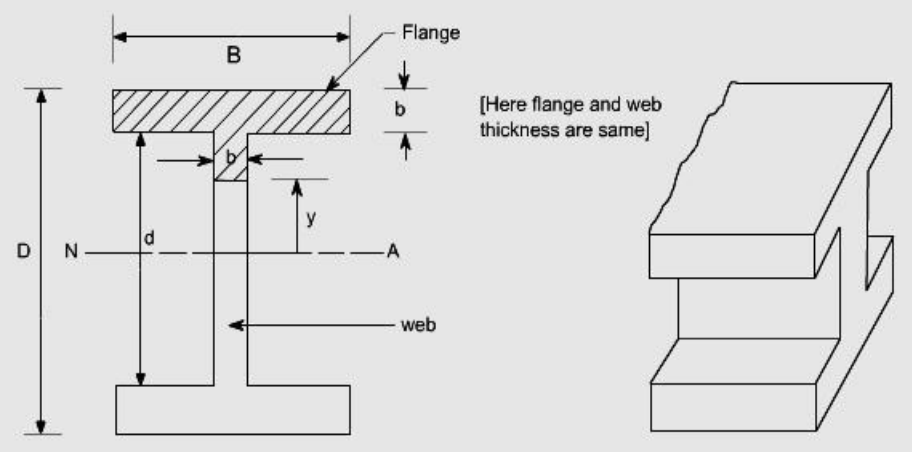
τ_{mean} or $\tau_{avg} = F/A = F/b.d$

So $\tau_{max} = 1.5\ \tau_{mean} = 1.5\ \tau_{avg}$

Therefore the shear stress distribution is shown as below.
It may be noted that the shear stress is distributed parabolically over a rectangular cross-section, it is maximum at y = 0 and is zero at the extreme ends.



I –section : Consider an **I** - section of the dimension shown below.

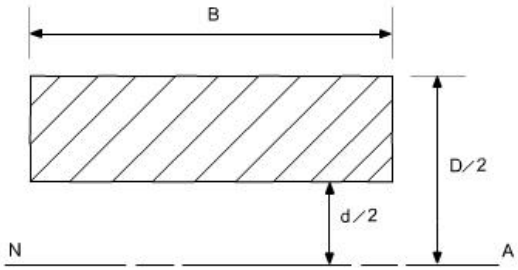


The shear stress distribution for any arbitrary shape is given as

$$\tau = \frac{F\,A\,\bar{y}}{\bar{Z}\,I}$$

Let us evaluate the quantity $A\bar{y}$, the quantity $A\bar{y}$ for this case comprise the contribution due to flange area and web area.

Flange area



$$\text{Area of the flange} = B\left(\frac{D-d}{2}\right)$$

Distance of the centroid of the flange from the N.A

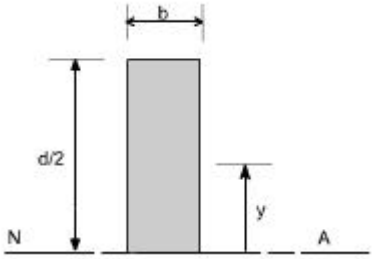
$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\big|_{Flange} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$

Web Area



Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

$$A\bar{y}\big|_{Total} = B\left(\frac{D-d}{2}\right)\left(\frac{D+d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

$$A\bar{y}\big|_{Total} = B\left(\frac{D^2 - d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2 - d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

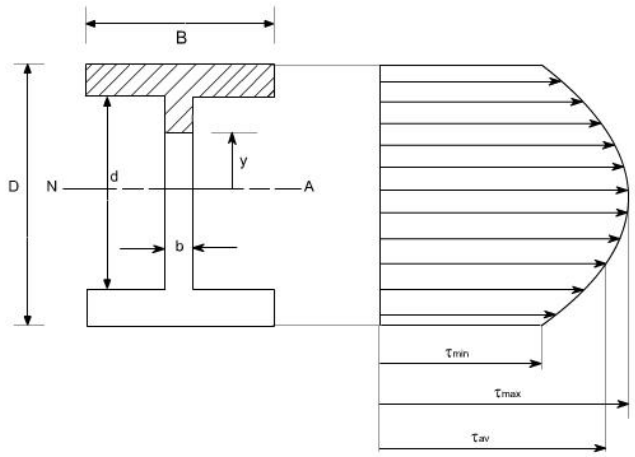
To get the maximum and minimum values of t substitute in the above relation.
y = 0 at N. A. And y = d/2 at the tip.
The maximum shear stress is at the neutral axis. i.e. for the condition y = 0 at N. A.

$$\tau_{max} \text{ at } y=0=\frac{F}{8bI}\left[B\left(D^2-d^2\right)+bd^2\right] \quad .....(2)$$

Hence,
The minimum stress occur at the top of the web, the term bd 2 goes off and shear stress is given by the following expression :

$$\tau_{min} \text{ at } y=d/2=\frac{F}{8bI}\left[B\left(D^2-d^2\right)\right] \quad .....(3)$$

The distribution of shear stress may be drawn as below, which clearly indicates a parabolic distribution:

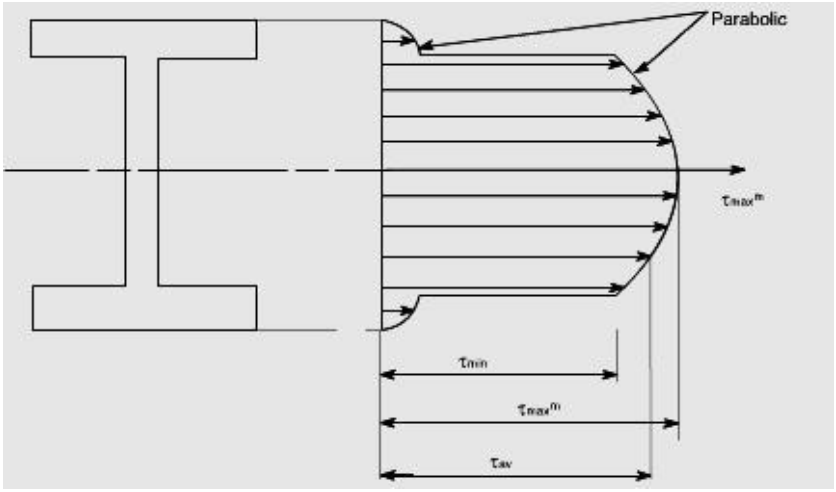


$$\tau_{max^m}=\frac{F}{8bI}\left[B\left(D^2-d^2\right)+bd^2\right]$$

Note: from the above distribution we can see that the shear stress at the flanges is not zero, but it has some value, this can be analyzed from equation (1). At the flange tip or flange or web interface y = d/2. Obviously then this will have some constant value and then onwards this will have parabolic distribution.

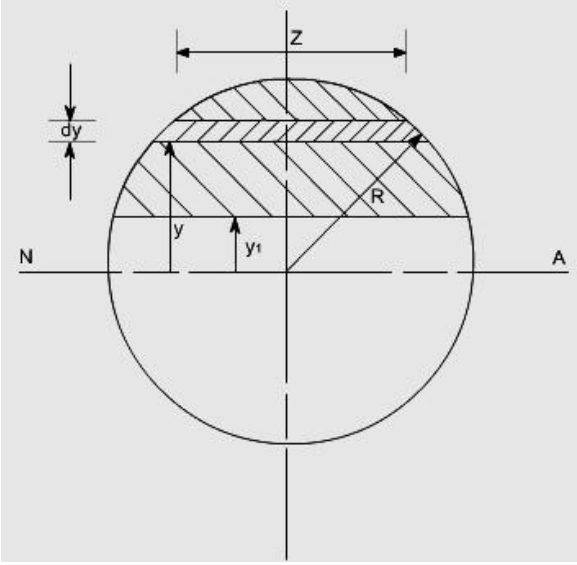
In practice it is usually found that most of shearing stress usually about 95% is carried by the web, and hence the shear stress in the flange is negligible however if we have the concrete analysis i.e. if we analyze the shearing stress in the flange i.e. writing down the expression for shear stress for flange and web separately, we will have this type of variation:

This distribution is known as the "top – hat" distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.



Shear stress distribution in beams of circular cross-section:

Let us find the shear stress distribution in beams of circular cross-section. In a beam of circular cross-section, the value of Z width depends on y.



Using the expression for the determination of shear stresses for any arbitrary shape or a arbitrary section.

$$\tau = \frac{FA\,\bar{y}}{Z\,I} = \frac{FA\int y\,dA}{Z\,I}$$

Where òy dA is the area moment of the shaded portion or the first moment of area.
Here in this case 'dA' is to be found out using the Pythagoras theorem:

$$\left(\frac{Z}{2}\right)^2 + y^2 = R^2$$

$$\left(\frac{Z}{2}\right)^2 = R^2 - y^2 \text{ or } \frac{Z}{2} = \sqrt{R^2 - y^2}$$

$$Z = 2\sqrt{R^2 - y^2}$$

$$dA = Z\,dy = 2.\sqrt{R^2 - y^2}\,.dy$$

$$I_{N.A \text{ for a circular cross-section}} = \frac{\pi R^4}{4}$$

Hence,

$$\tau = \frac{FA\,\bar{y}}{Z\,I} = \frac{F}{\frac{\pi R^4}{4}\,2\sqrt{R^2 - y^2}} \int_{y_1}^{R} 2\,y\sqrt{R^2 - y^2}\,dy$$

Where R = radius of the circle.
[The limits have been taken from y_1 to R because we have to find moment of area the shaded portion]

$$= \frac{4F}{\pi R^4\sqrt{R^2 - y^2}} \int_{y_1}^{R} y\sqrt{R^2 - y^2}\,dy$$

The integration yields the final result to be

$$\tau = \frac{4F\left(R^2 - y_1^2\right)}{3\pi R^4}$$

Again this is a parabolic distribution of shear stress, having a maximum value when $y_1 = 0$

$$\tau_{max^m}\,|\,y_1 = 0 = \frac{4F}{3\pi R^2}$$

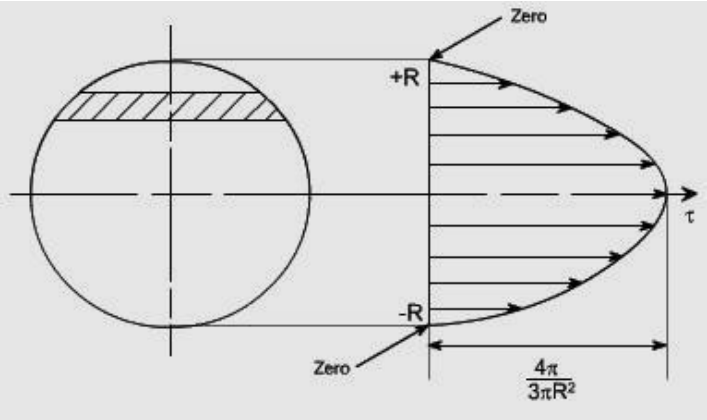
Obviously at the ends of the diameter the value of $y_1 = \pm R$ thus $\tau = 0$
so this again a parabolic distribution; maximum at the neutral axis
Also

$$\tau_{avg} \text{ or } \tau_{mean} = \frac{F}{A} = \frac{F}{\pi R^2}$$

Hence,

$$\boxed{\tau_{max^m} = \frac{4}{3}\,\tau_{avg}}$$

The distribution of shear stresses is shown below, which indicates a parabolic distribution:

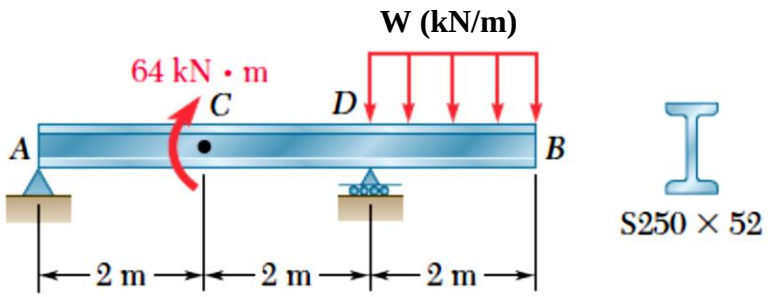


Quiz No. 4 :

For the beam and loading shown :

Draw the shear and bending-moment diagrams and determine the maximum value of (w) which can be applied such that the normal bending stress will not exceed (120 MPa).

For the section S250x52 use : $I = 61.2 \times 10^6\ mm^4$, $S = 486 \times 10^3\ mm^3$



Unsymmetrical Beams

Flexural Stress varies directly linearly with distance from the neutral axis. Thus for a symmetrical section such as wide flange, the compressive and tensile stresses will be the same. This will be desirable if the material is both equally strong in tension and compression. However, there are materials, such as cast iron, which are strong in compression than in tension. It is therefore desirable to use a beam with unsymmetrical cross section giving more area in the compression part making the stronger fiber located at a greater distance from the neutral axis than the weaker fiber. Some of these sections are shown below.

The proportioning of these sections is such that the ratio of the distance of the neutral axis from the outermost fibers in tension and in compression is the same as the ratio of the allowable stresses in tension and in compression. Thus, the allowable stresses are reached simultaneously.

In this section, the following notation will be use:

f_{bt} = *flexure stress of fiber in tension.*

f_{bc} = *flexure stress of fiber in compression.*

N.A. = neutral axis.

y_t = *distance of fiber in tension from N.A.*

y_c = *distance of fiber in compression from N.A.*

Mr = resisting moment.

Mc = resisting moment in compression.

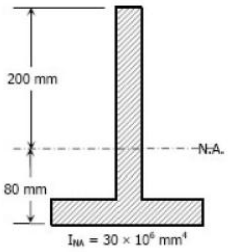
Mt = resisting moment in tension.

Solved Problems in Unsymmetrical Beams

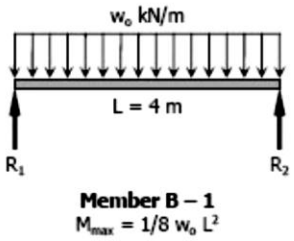
Example (1)

The inverted T- section of a 4-m simply supported beam has the properties shown in Fig. The beam carries a uniformly distributed load of intensity w_o over its entire length. Determine w_o if $\mathbf{f_{bt} \leq 40}$ MPa and $\mathbf{f_{bc} \leq 80}$ MPa.

Solution:



$$M_{max} = \tfrac{1}{8} w_o L^2$$
$$= \tfrac{1}{8} w_o (4^2)$$
$$= 2w_o$$



Member B – 1
$M_{max} = 1/8\ w_o\ L^2$

$$M_r = \frac{f_b I}{y}$$

$$M_t = \frac{40(30\times10^6)}{80}$$
$$= 15\,000\,000 \text{ N·mm}$$
$$= 15 \text{ kN·mm}$$

$$M_c = \frac{80(30\times10^6)}{200}$$
$$= 12\,000\,000 \text{ N·mm}$$
$$= 12 \text{ kN·mm}$$

The section is stronger in tension and weaker in compression, so compression governs in selecting the maximum moment.

$$M_{max} = M_r$$
$$2w_o = 12$$
$$w_o = \mathbf{6 \text{ kN/m}}$$

Example (2)

Determine the maximum tensile and compressive stresses developed in the overhanging beam shown in Fig. P-554. The cross-section is an inverted T with the given properties.

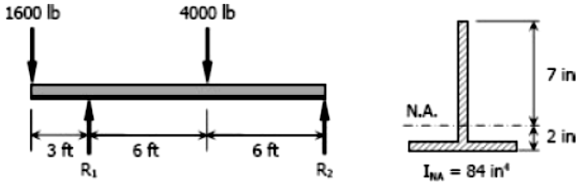
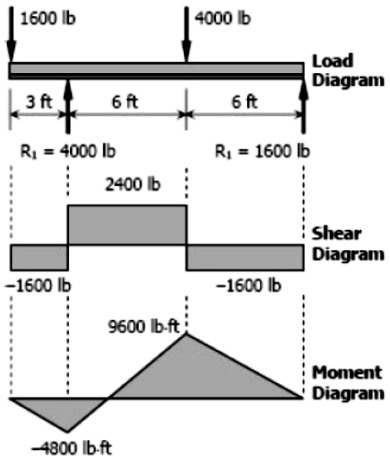


Figure P-554

Solution 554

$\Sigma M_{R2} = 0$
$12R_1 = 1600(15) + 4000(6)$
$R_1 = 4000$ lb

$\Sigma M_{R1} = 0$
$12R_2 + 1600(3) = 4000(6)$
$R_2 = 1600$ lb



$$f_b = \frac{My}{I}$$

At $M = -4800$ lb·ft

$$f_{bc} = \frac{4800(2)(12)}{84} = 1371.43 \text{ psi} \rightarrow \text{lower fiber}$$

$$f_{bt} = \frac{4800(7)(12)}{84} = 4800 \text{ psi} \rightarrow \text{upper fiber}$$

At $M = +9600$ lb·ft

$$f_{bc} = \frac{9600(7)(12)}{84} = 9600 \text{ psi} \rightarrow \text{upper fiber}$$

$$f_{bt} = \frac{9600(2)(12)}{84} = 2742.86 \text{ psi} \rightarrow \text{lower fiber}$$

Maximum flexure stress:
$f_{bc} = 9600$ psi
$f_{bt} = 4800$ psi

Example (3)

A cantilever beam carries the force and couple shown in Fig. P-552. Determine the maximum tensile and compressive bending stresses developed in the beam.

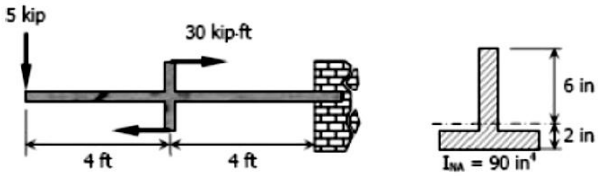
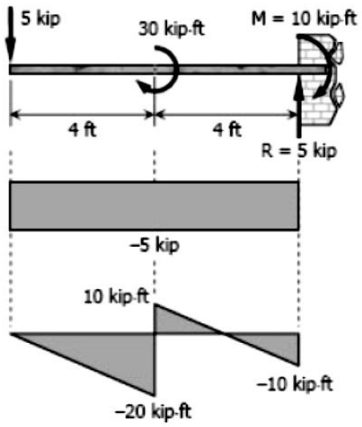


Figure P-552

Solution 552



$R = 5$ kip

$M = 5(8) - 30$

$= 10$ kip·ft

$$f_b = \frac{My}{I}$$

At $M = +10$ kip·ft of moment diagram

$$f_{bc} = \frac{10(6)(12)}{90}$$

$= 8$ ksi → upper fiber

$$f_{bt} = \frac{10(2)(12)}{90}$$

$= 2.67$ ksi → lower fiber

At $M = -20$ kip·ft of moment diagram

$$f_{bc} = \frac{20(2)(12)}{90}$$

$= 5.33$ ksi → lower fiber

$$f_{bt} = \frac{20(6)(12)}{90}$$

$= 16$ ksi → upper fiber

Maximum bending stresses:

$f_{bc} = 8$ ksi

$f_{bt} = 16$ ksi

Example (4)

A beam carries a concentrated load W and a total uniformly distributed load of 4W as shown in Fig. P-555. What safe value of W can be applied if $f_{bc} \le 100$ MPa and $f_{bt} \le 60$ MPa? Can a greater load be applied if the section is inverted? Explain.

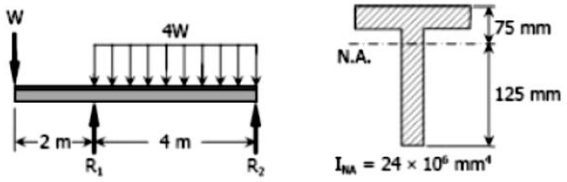
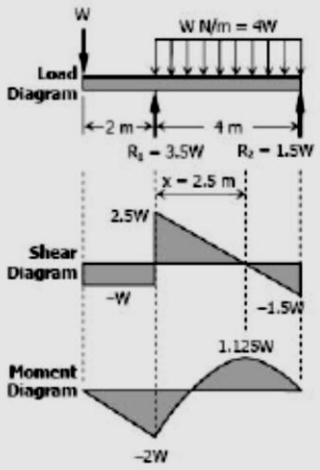


Figure P-555



$[\Sigma M_{R2} = 0]$ $\quad 4R_1 = 6W + 4W(2)$

$R_1 = 3.5W$

$[\Sigma M_{R1} = 0]$ $\quad 4R_2 + 2W = 4W(2)$

$R_2 = 1.5W$

$$\frac{x}{2.5W} = \frac{4-x}{1.5W}$$

$1.5Wx = 10W - 2.5Wx$

$x = 2.5$ m

$$f_b = \frac{My}{I}$$

At $M = -2W$

For lower fiber, $f_{bc} \le 100$ MPa

$$100 = \frac{2W(125)(1000)}{24 \times 10^6}$$

$W = 9600$ N

For upper fiber, $f_{bt} \le 60$ MPa

$$60 = \frac{2W(75)(1000)}{24 \times 10^6}$$

$W = 9600$ N

At $M = 1.125W$

For upper fiber, $f_{bc} \le 100$ MPa

$$100 = \frac{1.125W(75)(1000)}{24 \times 10^6}$$

$W = 28{,}444.44$ N

For lower fiber, $f_{bt} \le 60$ MPa

$$60 = \frac{1.125W(125)(1000)}{24 \times 10^6}$$

$W = 10{,}240$ N

For safe load W, use $W = 9600$ N

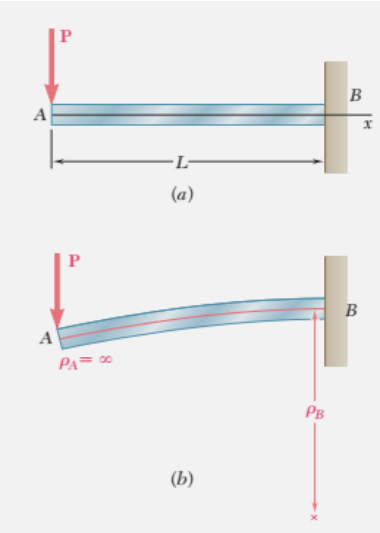
CHAPTER 7

Deflection of Beams

(I) Method of Double Integration

Differential Equations of the Deflection Curve (Elastic Curve):

The problem of bending probably occurs more often than any other loading problem in design. Shafts, axles, cranks, levers, springs, brackets, and wheels, as well as many other elements, must often be treated as beams in the design and analysis of mechanical structures and system.



A beam subjected to pure bending is bent into an arc of circle within the elastic range, and the relation for the curvature is:

$$\frac{1}{\rho} = \frac{M(x)}{EI} \qquad (1)$$

Where: ρ is the radius of the curvature of the neutral axis?

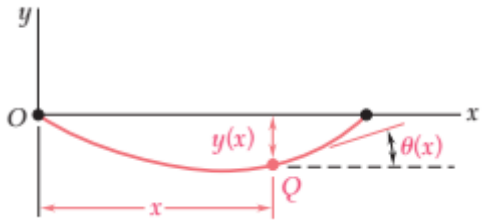
x is the distance of the section from the left end of the beam.

The curvature of a plane curve is given by the equation:

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \qquad (2)$$

$\left(\frac{dy}{dx}\right)$ is the slope of the curve and in the case of elastic curve the slope is very small:

$$\left(\frac{dy}{dx}\right)^2 = 0$$



then $$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

or $$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \qquad (3)$$

Multiply both sides by EI which is constant and integrating with respect to x:

$$EI\left(\frac{dy}{dx}\right) = \int M(x)\,dx + C_1 \qquad (4)$$

Noting that $\left(\frac{dy}{dx}\right) = \tan(\theta) = \theta(x)$ because the angle θ is very small.

Then Eq. (4) can be written as:

$$EI\,\theta(x) = \int_0^x M(x)\,dx + C_1$$

And integrating the equation again:

$$EI\,y = \int [\int M(x)dx + C_1]\,dx + C_2 \qquad (5)$$

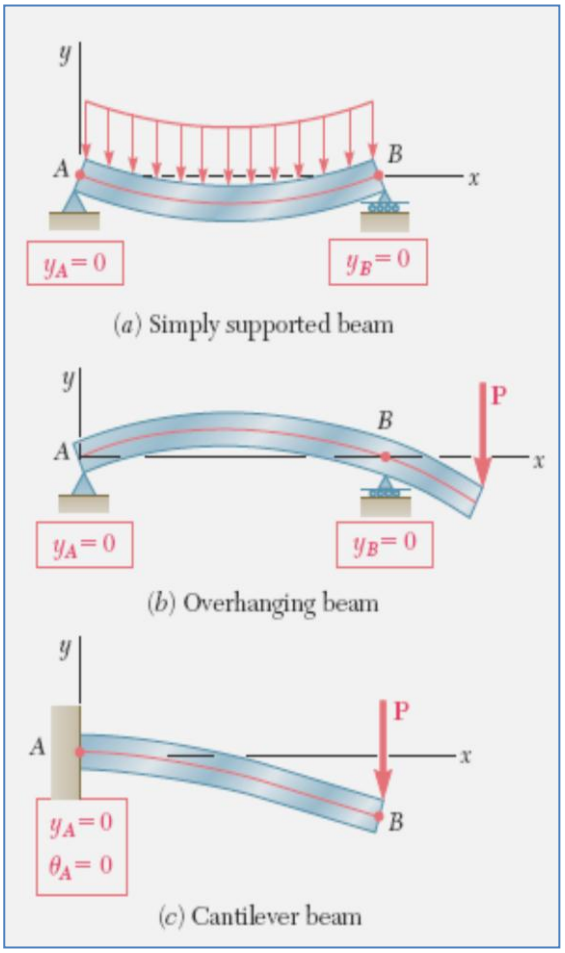
$$EI\,y = \int [\int M(x)dx]\,dx + C_1\,x + C_2$$

The constants C_1 and C_2 are determined from the boundary conditions (constants) imposed on the beam by its supports.

The figure shows different boundary conditions applied for the three typical types of statically determinate beams: (*a*) the *simply supported beam*, (*b*) the *overhanging beam*, and (*c*) the *cantilever beam*.

In the first two cases, the supports consist of a pin and bracket at **A** and of a roller at **B**, and require that the deflection be zero at each of these points. Letting first $x = x_A$, $y = y_A = 0$ in Eq. (5), and then $x = x_B$, $y = y_B = 0$ in the same equation, we obtain two equations that can be solved for $C1$ and $C2$.

In the case of the cantilever beam, we note that both the deflection and the slope at **A** must be zero. Letting $x = x_A$, $y = y_A = 0$ in Eq. (5), and $x = x_A$, $\theta = \theta_A = 0$ in Eq. (4), we obtain again two equations that can be solved for $C1$ and $C2$.



EXAMPLE 9.01

The cantilever beam AB is of uniform cross section and carries a load **P** at its free end A (Fig. 9.9). Determine the equation of the elastic curve and the deflection and slope at A.

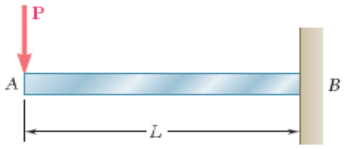


Fig. 9.9

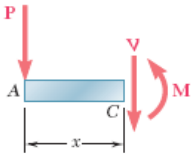


Fig. 9.10

Using the free-body diagram of the portion AC of the beam (Fig. 9.10), where C is located at a distance x from end A, we find

$$M = -Px \tag{9.7}$$

Substituting for M into Eq. (9.4) and multiplying both members by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -Px$$

Integrating in x, we obtain

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + C_1 \tag{9.8}$$

We now observe that at the fixed end B we have $x = L$ and $\theta = dy/dx = 0$ (Fig. 9.11). Substituting these values into (9.8) and solving for C_1, we have

$$C_1 = \tfrac{1}{2}PL^2$$

which we carry back into (9.8):

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + \tfrac{1}{2}PL^2 \tag{9.9}$$

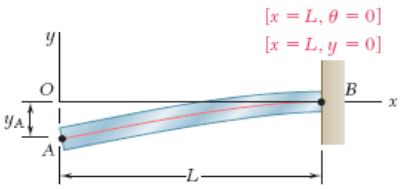


Fig. 9.11

Integrating both members of Eq. (9.9), we write

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x + C_2 \tag{9.10}$$

But, at B we have $x = L$, $y = 0$. Substituting into (9.10), we have

$$0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^3 + C_2$$
$$C_2 = -\tfrac{1}{3}PL^3$$

Carrying the value of C_2 back into Eq. (9.10), we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3$$

or

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \tag{9.11}$$

The deflection and slope at A are obtained by letting $x = 0$ in Eqs. (9.11) and (9.9). We find

$$y_A = -\frac{PL^3}{3EI} \qquad \text{and} \qquad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$

EXAMPLE 9.02

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 9.12). Determine the equation of the elastic curve and the maximum deflection of the beam.

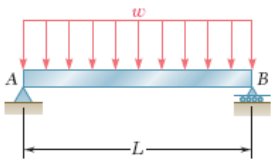


Fig. 9.12

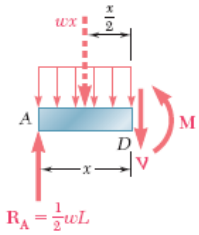


Fig. 9.13

Drawing the free-body diagram of the portion AD of the beam (Fig. 9.13) and taking moments about D, we find that

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2 \tag{9.12}$$

Substituting for M into Eq. (9.4) and multiplying both members of this equation by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx \tag{9.13}$$

Integrating twice in x, we have

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1 \tag{9.14}$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2 \tag{9.15}$$

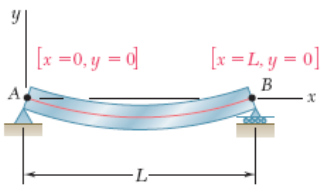


Fig. 9.14

Observing that $y = 0$ at both ends of the beam (Fig. 9.14), we first let $x = 0$ and $y = 0$ in Eq. (9.15) and obtain $C_2 = 0$. We then make $x = L$ and $y = 0$ in the same equation and write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_1L$$

$$C_1 = -\tfrac{1}{24}wL^3$$

Carrying the values of C_1 and C_2 back into Eq. (9.15), we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wLx^3 - \tfrac{1}{24}wL^3x$$

or

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \tag{9.16}$$

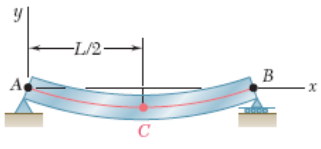


Fig. 9.15

Substituting into Eq. (9.14) the value obtained for C_1, we check that the slope of the beam is zero for $x = L/2$ and that the elastic curve has a minimum at the midpoint C of the beam (Fig. 9.15). Letting $x = L/2$ in Eq. (9.16), we have

$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

The maximum deflection or, more precisely, the maximum absolute value of the deflection, is thus

$$|y|_{\max} = \frac{5wL^4}{384EI}$$

In each of the two examples considered so far, only one free body diagram was required to determine the bending moment in the beam. As a result, a single function of ***x*** was used to represent ***M*** throughout the beam. This, however, is not generally the case. Concentrated loads, reactions at supports, or discontinuities in a distributed load will make it necessary to divide the beam into several portions, and to represent the bending moment by a different function ***M*(*x*)** in each of these portions of beam. Each of the functions ***M*(*x*)** will then lead to a different expression for the slope **θ(*x*)** and for the deflection ***y*(*x*)**. Since each of the expressions obtained for the deflection must contain two constants of integration, a large number of constants will have to be determined.

As you will see in the next example, the required additional boundary conditions can be obtained by observing that, while the shear and bending moment can be discontinuous at several points in a beam, the ***deflection*** and the ***slope*** of the beam *cannot be discontinuous* at any point.

EXAMPLE 9.03

For the prismatic beam and the loading shown (Fig. 9.16), determine the slope and deflection at point D.

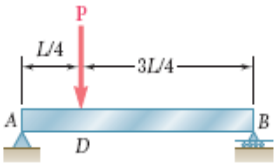


Fig. 9.16

We must divide the beam into two portions, AD and DB, and determine the function $y(x)$ which defines the elastic curve for each of these portions.

1. From *A* to *D* ($x < L/4$). We draw the free-body diagram of a portion of beam AE of length $x < L/4$ (Fig. 9.17). Taking moments about E, we have

$$M_1 = \frac{3P}{4}x \tag{9.17}$$

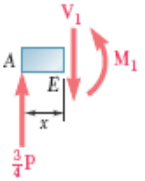


Fig. 9.17

or, recalling Eq. (9.4),

$$EI\frac{d^2y_1}{dx^2} = \frac{3}{4}Px \tag{9.18}$$

where $y_1(x)$ is the function which defines the elastic curve *for portion AD of the beam*. Integrating in x, we write

$$EI\,\theta_1 = EI\frac{dy_1}{dx} = \frac{3}{8}Px^2 + C_1 \tag{9.19}$$

$$EI\,y_1 = \frac{1}{8}Px^3 + C_1x + C_2 \tag{9.20}$$

2. From *D* to *B* ($x > L/4$). We now draw the free-body diagram of a portion of beam AE of length $x > L/4$ (Fig. 9.18) and write

$$M_2 = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right) \tag{9.21}$$

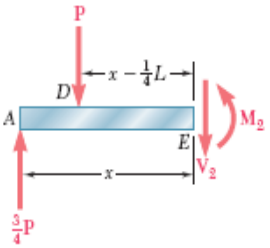


Fig. 9.18

or, recalling Eq. (9.4) and rearranging terms,

$$EI\frac{d^2y_2}{dx^2} = -\frac{1}{4}Px + \frac{1}{4}PL \tag{9.22}$$

where $y_2(x)$ is the function which defines the elastic curve *for portion DB of the beam*. Integrating in x, we write

$$EI\,\theta_2 = EI\frac{dy_2}{dx} = -\frac{1}{8}Px^2 + \frac{1}{4}PLx + C_3 \tag{9.23}$$

$$EI\,y_2 = -\frac{1}{24}Px^3 + \frac{1}{8}PLx^2 + C_3x + C_4 \tag{9.24}$$

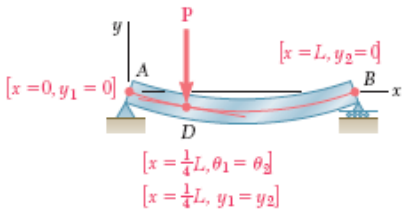


Fig. 9.19

Determination of the Constants of Integration. The conditions that must be satisfied by the constants of integration have been summarized in Fig. 9.19. At the support A, where the deflection is defined by Eq. (9.20), we must have $x = 0$ and $y_1 = 0$. At the support B, where the deflection is defined by Eq. (9.24), we must have $x = L$ and $y_2 = 0$. Also, the fact that there can be no sudden change in deflection or in slope at point D requires that $y_1 = y_2$ and $\theta_1 = \theta_2$ when $x = L/4$. We have therefore:

$[x = 0, y_1 = 0]$, Eq. (9.20): $\quad 0 = C_2$ (9.25)

$[x = L, y_2 = 0]$, Eq. (9.24): $\quad 0 = \frac{1}{12}PL^3 + C_3L + C_4$ (9.26)

$[x = L/4, \theta_1 = \theta_2]$, Eqs. (9.19) and (9.23):

$$\frac{3}{128}PL^2 + C_1 = \frac{7}{128}PL^2 + C_3 \tag{9.27}$$

$[x = L/4, y_1 = y_2]$, Eqs. (9.20) and (9.24):

$$\frac{PL^3}{512} + C_1\frac{L}{4} = \frac{11PL^3}{1536} + C_3\frac{L}{4} + C_4 \tag{9.28}$$

Solving these equations simultaneously, we find

$$C_1 = -\frac{7PL^2}{128}, C_2 = 0, C_3 = -\frac{11PL^2}{128}, C_4 = \frac{PL^3}{384}$$

Substituting for C_1 and C_2 into Eqs. (9.19) and (9.20), we write that for $x \le L/4$,

$$EI\,\theta_1 = \frac{3}{8}Px^2 - \frac{7PL^2}{128} \tag{9.29}$$

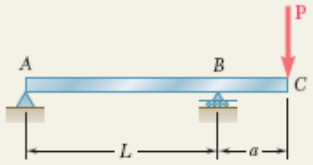
$$EI\,y_1 = \frac{1}{8}Px^3 - \frac{7PL^2}{128}x \tag{9.30}$$

Letting $x = L/4$ in each of these equations, we find that the slope and deflection at point D are, respectively,

$$\theta_D = -\frac{PL^2}{32EI} \quad \text{and} \quad y_D = -\frac{3PL^3}{256EI}$$

We note that, since $\theta_D \neq 0$, the deflection at D is *not* the maximum deflection of the beam.

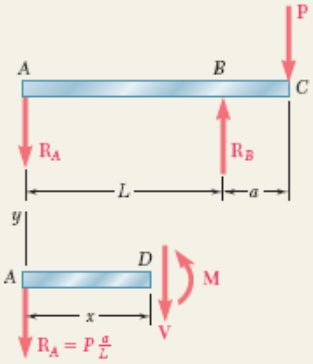
SAMPLE PROBLEM 9.1



The overhanging steel beam *ABC* carries a concentrated load **P** at end *C*. For portion *AB* of the beam, (*a*) derive the equation of the elastic curve, (*b*) determine the maximum deflection, (*c*) evaluate y_{max} for the following data:

W14 × 68	$I = 722\ \text{in}^4$	$E = 29 \times 10^6\ \text{psi}$
$P = 50$ kips	$L = 15\ \text{ft} = 180\ \text{in.}$	$a = 4\ \text{ft} = 48\ \text{in.}$

SOLUTION



Free-Body Diagrams. Reactions: $\mathbf{R}_A = Pa/L \downarrow$ $\mathbf{R}_B = P(1 + a/L) \uparrow$ Using the free-body diagram of the portion of beam *AD* of length *x*, we find

$$M = -P\frac{a}{L}x \qquad (0 < x < L)$$

Differential Equation of the Elastic Curve. We use Eq. (9.4) and write

$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

Noting that the flexural rigidity *EI* is constant, we integrate twice and find

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1 \tag{1}$$

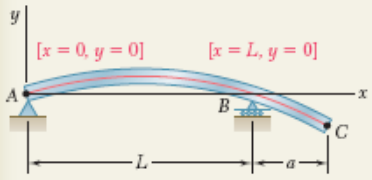
$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2 \tag{2}$$

Determination of Constants. For the boundary conditions shown, we have

$[x = 0, y = 0]$: From Eq. (2), we find $C_2 = 0$

$[x = L, y = 0]$: Again using Eq. (2), we write

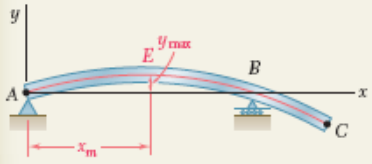
$$EI(0) = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \qquad C_1 = +\frac{1}{6}PaL$$



a. Equation of the Elastic Curve. Substituting for C_1 and C_2 into Eqs. (1) and (2), we have

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \qquad \frac{dy}{dx} = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right] \tag{3}$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx \qquad y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right] \tag{4}$$



b. Maximum Deflection in Portion *AB*. The maximum deflection y_{max} occurs at point *E* where the slope of the elastic curve is zero. Setting $dy/dx = 0$ in Eq. (3), we determine the abscissa x_m of point *E*:

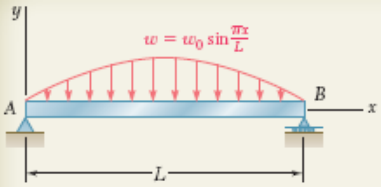
$$0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

We substitute $x_m/L = 0.577$ into Eq. (4) and have

$$y_{max} = \frac{PaL^2}{6EI}[(0.577) - (0.577)^3] \qquad y_{max} = 0.0642\frac{PaL^2}{EI}$$

c. Evaluation of y_{max}. For the data given, the value of y_{max} is

$$y_{max} = 0.0642\frac{(50\ \text{kips})(48\ \text{in.})(180\ \text{in.})^2}{(29 \times 10^6\ \text{psi})(722\ \text{in}^4)} \qquad y_{max} = 0.238\ \text{in.}$$



SAMPLE PROBLEM 9.2

For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at end *A*, (*c*) the maximum deflection.

SOLUTION

Differential Equation of the Elastic Curve. From Eq. (9.32),

$$EI\frac{d^4y}{dx^4} = -w(x) = -w_0 \sin\frac{\pi x}{L} \tag{1}$$

Integrate Eq. (1) twice:

$$EI\frac{d^3y}{dx^3} = V = +w_0\frac{L}{\pi}\cos\frac{\pi x}{L} + C_1 \tag{2}$$

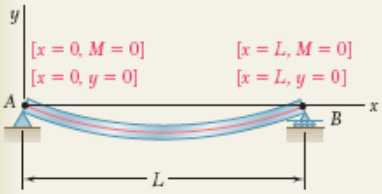
$$EI\frac{d^2y}{dx^2} = M = +w_0\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} + C_1x + C_2 \tag{3}$$

Boundary Conditions:

$[x = 0, M = 0]$: From Eq. (3), we find $C_2 = 0$

$[x = L, M = 0]$: Again using Eq. (3), we write

$$0 = w_0\frac{L^2}{\pi^2}\sin\pi + C_1L \quad C_1 = 0$$



Thus:

$$EI\frac{d^2y}{dx^2} = +w_0\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} \tag{4}$$

Integrate Eq. (4) twice:

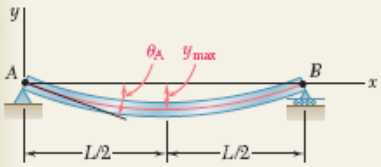
$$EI\frac{dy}{dx} = EI\,\theta = -w_0\frac{L^3}{\pi^3}\cos\frac{\pi x}{L} + C_3 \tag{5}$$

$$EI\,y = -w_0\frac{L^4}{\pi^4}\sin\frac{\pi x}{L} + C_3x + C_4 \tag{6}$$

Boundary Conditions:

$[x = 0, y = 0]$: Using Eq. (6), we find $C_4 = 0$

$[x = L, y = 0]$: Again using Eq. (6), we find $C_3 = 0$



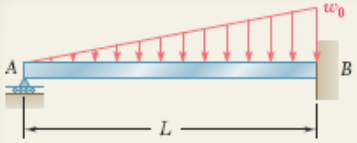
a. Equation of Elastic Curve $EIy = -w_0\frac{L^4}{\pi^4}\sin\frac{\pi x}{L}$ ◀

b. Slope at End A. For $x = 0$, we have

$$EI\,\theta_A = -w_0\frac{L^3}{\pi^3}\cos 0 \qquad \theta_A = \frac{w_0L^3}{\pi^3EI} \curvearrowleft$$ ◀

c. Maximum Deflection. For $x = \frac{1}{2}L$

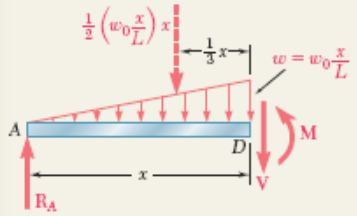
$$EIy_{max} = -w_0\frac{L^4}{\pi^4}\sin\frac{\pi}{2} \qquad y_{max} = \frac{w_0L^4}{\pi^4EI} \downarrow$$ ◀



SAMPLE PROBLEM 9.3

For the uniform beam AB, (*a*) determine the reaction at A, (*b*) derive the equation of the elastic curve, (*c*) determine the slope at A. (Note that the beam is statically indeterminate to the first degree.)

SOLUTION



Bending Moment. Using the free body shown, we write

$$+\circlearrowleft \Sigma M_D = 0: \qquad R_A x - \frac{1}{2}\left(\frac{w_0 x^2}{L}\right)\frac{x}{3} - M = 0 \qquad M = R_A x - \frac{w_0 x^3}{6L}$$

Differential Equation of the Elastic Curve. We use Eq. (9.4) and write

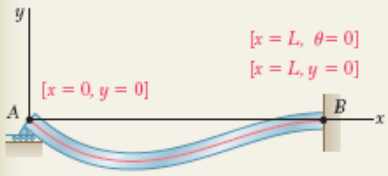
$$EI\frac{d^2y}{dx^2} = R_A x - \frac{w_0 x^3}{6L}$$

Noting that the flexural rigidity EI is constant, we integrate twice and find

$$EI\frac{dy}{dx} = EI\,\theta = \frac{1}{2}R_A x^2 - \frac{w_0 x^4}{24L} + C_1 \tag{1}$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{w_0 x^5}{120L} + C_1 x + C_2 \tag{2}$$

Boundary Conditions. The three boundary conditions that must be satisfied are shown on the sketch



$$[x = 0, y = 0]: \quad C_2 = 0 \tag{3}$$

$$[x = L, \theta = 0]: \quad \frac{1}{2}R_A L^2 - \frac{w_0 L^3}{24} + C_1 = 0 \tag{4}$$

$$[x = L, y = 0]: \quad \frac{1}{6}R_A L^3 - \frac{w_0 L^4}{120} + C_1 L + C_2 = 0 \tag{5}$$

a. Reaction at A. Multiplying Eq. (4) by L, subtracting Eq. (5) member by member from the equation obtained, and noting that $C_2 = 0$, we have

$$\tfrac{1}{3}R_A L^3 - \tfrac{1}{30}w_0 L^4 = 0 \qquad R_A = \tfrac{1}{10}w_0 L \uparrow$$

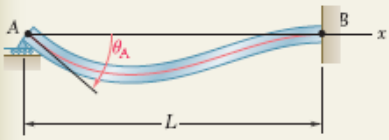
We note that the reaction is independent of E and I. Substituting $R_A = \frac{1}{10}w_0 L$ into Eq. (4), we have

$$\tfrac{1}{2}(\tfrac{1}{10}w_0 L)L^2 - \tfrac{1}{24}w_0 L^3 + C_1 = 0 \qquad C_1 = -\tfrac{1}{120}w_0 L^3$$

b. Equation of the Elastic Curve. Substituting for R_A, C_1, and C_2 into Eq. (2), we have

$$EI\,y = \frac{1}{6}\left(\frac{1}{10}w_0 L\right)x^3 - \frac{w_0 x^5}{120L} - \left(\frac{1}{120}w_0 L^3\right)x$$

$$y = \frac{w_0}{120EIL}(-x^5 + 2L^2x^3 - L^4x)$$



c. Slope at A. We differentiate the above equation with respect to x:

$$\theta = \frac{dy}{dx} = \frac{w_0}{120EIL}(-5x^4 + 6L^2x^2 - L^4)$$

Making $x = 0$, we have $\qquad \theta_A = -\dfrac{w_0 L^3}{120EI} \qquad \theta_A = \dfrac{w_0 L^3}{120EI}\ \curvearrowleft$

USING SINGULARITY FUNCTIONS TO DETERMINE THE SLOPE AND DEFLECTION OF A BEAM

Let us consider again the beam and loading of Example 9.03 (Fig. 9.16) and draw the free-body diagram of that beam (Fig. 9.27). Using the appropriate singularity function, to represent the contribution to the shear of the concentrated load **P,** we write:

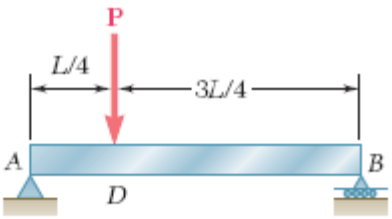


Fig. 9.16 *(repeated)*

$$V(x) = \frac{3P}{4} - P\langle x - \tfrac{1}{4}L\rangle^0$$

Integrating in x and recalling that in the absence of any concentrated couple, the expression obtained for the bending moment will not contain any constant term, we have:

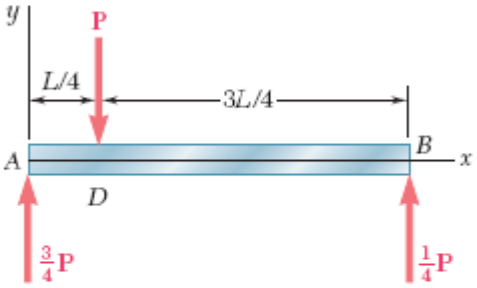


Fig. 9.27 Free-body diagram for beam of Fig. 9.16.

$$M(x) = \frac{3P}{4}x - P\langle x - \tfrac{1}{4}L\rangle$$

$$EI\frac{d^2y}{dx^2} = \frac{3P}{4}x - P\langle x - \tfrac{1}{4}L\rangle$$

and, integrating in x,

$$EI\,\theta = EI\frac{dy}{dx} = \frac{3}{8}Px^2 - \frac{1}{2}P\langle x - \tfrac{1}{4}L\rangle^2 + C_1$$

$$EI\,y = \frac{1}{8}Px^3 - \frac{1}{6}P\langle x - \tfrac{1}{4}L\rangle^3 + C_1x + C_2$$

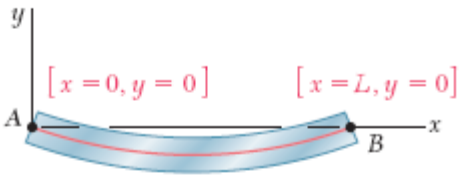


Fig. 9.28 Boundary conditions for beam of Fig. 9.16.

The constants C_1 and C_2 can be determined from the boundary conditions shown in Fig. 9.28. Letting $x = 0$, $y = 0$ in Eq. of y above , we have:

$$0 = 0 - \frac{1}{6}P\langle 0 - \tfrac{1}{4}L\rangle^3 + 0 + C_2$$

which reduces to $C_2 = 0$, since any bracket containing a negative quantity is equal to zero.

Letting now $x = L$, $y = 0$, and $C_2 = 0$ in the same equation of y above, we write:

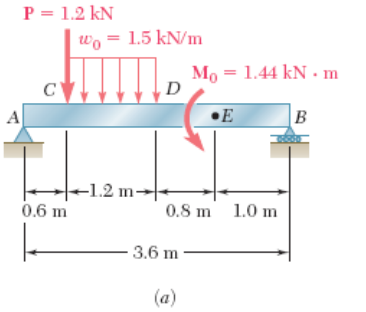
$$0 = \frac{1}{8}PL^3 - \frac{1}{6}P\langle\tfrac{3}{4}L\rangle^3 + C_1L$$

Since the quantity between brackets is positive, the brackets can be replaced by ordinary parentheses. Solving for $C1$, we have:

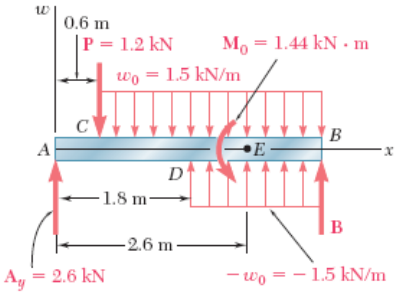
$$C_1 = -\frac{7PL^2}{128}$$

EXAMPLE 9.06

For the beam and loading shown (Fig. 9.29*a*) and using singularity functions, (*a*) express the slope and deflection as functions of the distance x from the support at A, (*b*) determine the deflection at the midpoint D. Use $E = 200$ GPa and $I = 6.87 \times 10^{-6}$ m^4.



(*a*)



(*b*)

Fig. 9.29

(*a*) We note that the beam is loaded and supported in the same manner as the beam of Example 5.05. Referring to that example, we recall that the given distributed loading was replaced by the two equivalent open-ended loadings shown in Fig. 9.29*b* and that the following expressions were obtained for the shear and bending moment:

$$V(x) = -1.5\langle x - 0.6\rangle^1 + 1.5\langle x - 1.8\rangle^1 + 2.6 - 1.2\langle x - 0.6\rangle^0$$
$$M(x) = -0.75\langle x - 0.6\rangle^2 + 0.75\langle x - 1.8\rangle^2 + 2.6x - 1.2\langle x - 0.6\rangle^1 - 1.44\langle x - 2.6\rangle^0$$

Integrating the last expression twice, we obtain

$$EI\theta = -0.25\langle x - 0.6\rangle^3 + 0.25\langle x - 1.8\rangle^3 + 1.3x^2 - 0.6\langle x - 0.6\rangle^2 - 1.44\langle x - 2.6\rangle^1 + C_1 \qquad (9.48)$$
$$EIy = -0.0625\langle x - 0.6\rangle^4 + 0.0625\langle x - 1.8\rangle^4 + 0.4333x^3 - 0.2\langle x - 0.6\rangle^3 - 0.72\langle x - 2.6\rangle^2 + C_1x + C_2 \qquad (9.49)$$

The constants C_1 and C_2 can be determined from the boundary conditions shown in Fig. 9.30. Letting $x = 0$, $y = 0$ in Eq. (9.49) and noting that all the brackets contain negative quantities and, therefore, are equal to zero, we conclude that $C_2 = 0$. Letting now $x = 3.6$, $y = 0$, and $C_2 = 0$ in Eq. (9.49), we write

$$0 = -0.0625\langle 3.0\rangle^4 + 0.0625\langle 1.8\rangle^4 + 0.4333(3.6)^3 - 0.2\langle 3.0\rangle^3 - 0.72\langle 1.0\rangle^2 + C_1(3.6) + 0$$

Since all the quantities between brackets are positive, the brackets can be replaced by ordinary parentheses. Solving for C_1, we find $C_1 = -2.692$.

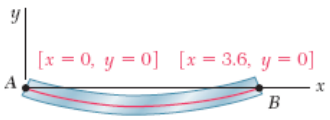


Fig. 9.30

(*b*) Substituting for C_1 and C_2 into Eq. (9.49) and making $x = x_D = 1.8$ m, we find that the deflection at point D is defined by the relation

$$EIy_D = -0.0625\langle 1.2\rangle^4 + 0.0625\langle 0\rangle^4 + 0.4333(1.8)^3 - 0.2\langle 1.2\rangle^3 - 0.72\langle -0.8\rangle^2 - 2.692(1.8)$$

The last bracket contains a negative quantity and, therefore, is equal to zero. All the other brackets contain positive quantities and can be replaced by ordinary parentheses. We have

$$EIy_D = -0.0625(1.2)^4 + 0.0625(0)^4 + 0.4333(1.8)^3 - 0.2(1.2)^3 - 0 - 2.692(1.8) = -2.794$$

Recalling the given numerical values of E and I, we write

$$(200 \text{ GPa})(6.87 \times 10^{-6} \text{ m}^4)y_D = -2.794 \text{ kN} \cdot \text{m}^3$$
$$y_D = -13.64 \times 10^{-3} \text{ m} = -2.03 \text{ mm}$$